

Hilbert series of modules over Lie algebroids

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Abstract

We consider modules M over Lie algebroids \mathfrak{g}_A which are of finite type over a local noetherian ring A . Using ideals $J \subset A$ such that $\mathfrak{g}_A \cdot J \subset J$ and the length $\ell_{\mathfrak{g}_A}(M/JM) < \infty$ we can define in a natural way the Hilbert series of M with respect to the defining ideal J . This notion is in particular studied for modules over the Lie algebroid of k -linear derivations $\mathfrak{g}_A = T_A(I)$ that preserve an ideal $I \subset A$, for example when $A = \mathcal{O}_n$, the ring of convergent power series.

Keywords: Hilbert series, tangential vector field, Lie algebroid, complete intersection, invariant rings, isolated singularity

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1. Introduction

Let (A, \mathfrak{m}_A, k) be an *allowed* local commutative noetherian k -algebra of characteristic 0, which entails in particular that the (generic) rank of the A -module of k -linear derivations $T_A := T_{A/k}$ coincides with the Krull dimension of A . Let I be an ideal of A and $T_A(I) \subset T_A$ be the A -submodule of k -linear derivations δ of A such that $\delta \cdot I \subset I$, which we call the tangential Lie algebroid along I . More generally, a Lie algebroid is an A -module \mathfrak{g}_A equipped with a structure of Lie algebra over k and a homomorphism of A -modules $\alpha : \mathfrak{g}_A \rightarrow T_A$ satisfying natural compatibility relations for the Lie algebra and module structures. The notion of module over a Lie algebroid \mathfrak{g}_A is what can be expected and is not explained here. If M is a \mathfrak{g}_A -module which is of finite type as A -module, we say that an ideal J is a *defining ideal* for M if $\alpha(\mathfrak{g}_A) \cdot J \subset J$ and the length $l_{\mathfrak{g}_A}(M/JM) < \infty$. We prove that if J is a defining ideal, then $l_{\mathfrak{g}_A}(J^n M/J^{n+1}M) < \infty$ for every positive integer n , so one can define the *Hilbert series of a \mathfrak{g}_A -module M with respect to the defining ideal J*

$$H_M^J(t) = \sum_{n \geq 0} l_{\mathfrak{g}_A}\left(\frac{J^n M}{J^{n+1}M}\right) t^n \in \mathbf{Z}[[t]].$$

This series extracts information about the complicated structure of \mathfrak{g}_A -modules, which in general do not have a finite length. Some natural examples of $T_A(I)$ -modules are A , I , the integral closure of I , and the Jacobian ideal of I [14]. In the study of regular singular \mathcal{D}_A -modules N , where \mathcal{D}_A is the ring of differential operators on A , there is a need to understand $T_A(I)$ -submodules $N_0 \subset N$ that are of finite type over A .

When $\alpha(\mathfrak{g}_A) = 0$, so \mathfrak{g}_A is a Lie algebra over A , a defining ideal for A is the same as \mathfrak{m}_A -primary ideal. If moreover our \mathfrak{g}_A -modules $\frac{J^n M}{J^{n+1}M}$ have complete flags, so $l_{\mathfrak{g}_A}(\frac{J^n M}{J^{n+1}M}) = \dim_k(\frac{J^n M}{J^{n+1}M})$, it follows from Hilbert's theorem that $H_M^J(t)$ is a rational function. If (R, \mathfrak{g}_R) is a Lie algebroid over an allowed regular local ring R , its *fibre Lie algebra* is $\mathfrak{g}_k = k \otimes_R \text{Ker}(\mathfrak{g}_R \rightarrow T_{R/k})$. For example, if I is a radical ideal in a regular allowed local ring A , defining a smooth variety of codimension r , and $\mathfrak{g}_A = T_A(I)$, then I is a maximal defining ideal of the $T_A(I)$ -modules A and the fibre Lie algebra \mathfrak{g}_k of $\mathfrak{g}_R = \mathfrak{g}_A/I\mathfrak{g}_A$ is isomorphic to $\mathfrak{gl}_r \oplus k^{\dim A - r}$, where \mathfrak{gl}_r is the general linear algebra and k is considered as a commutative Lie algebra. In general, the length of a \mathfrak{g}_R -module M of finite type over R is less than the length of the fibre $k \otimes_R M$ as module over the fibre Lie algebra \mathfrak{g}_k , but sometimes equality holds

$$\ell_{\mathfrak{g}_R}(M) = \ell_{\mathfrak{g}_k}(k \otimes_R M),$$

and we then say M is a *local system*. Letting J_m be a maximal defining ideal of the \mathfrak{g}_A -module A we prove that J_m is a maximal defining ideal of any \mathfrak{g}_A -module M of finite type over A , and that $R = A/J_m$ is a regular local ring. We can therefore more generally say M is a local system along the maximal defining ideal J_m if each homogeneous component of the graded \mathfrak{g}_R -module $G_{J_m}^\bullet(M) = \bigoplus_{i \geq 0} J_m^i M/J_m^{i+1}M$ is a local system.

Theorem 3.11. Let M be a \mathfrak{g}_A -module of finite type and J be a defining ideal. Assume that M is a local system along the radical of J . Then $H_M^J(t)$ is a rational function and the function $n \rightarrow \ell_{\mathfrak{g}_A}(M/J^{n+1}M)$ is a quasi-polynomial for high n .

The proof of Theorem 3.11 is based on a reduction to modules over the fibre Lie algebra \mathfrak{g}_k and taking invariants over a maximal nilpotent subalgebra of a Levi factor of \mathfrak{g}_k , applying Hilbert's finiteness theorem on the finite generation of invariant rings as extended by Hadziev [9], who used an idea that arguably can be traced back to the classical invariant theorists [23]. If M is not a local system, then the Hilbert series still is rational, but the proof requires a different approach, using a Tannakian theorem for \mathfrak{g}_A -modules in connection with a field extension to make our \mathfrak{g}_A -modules into local systems. This will be treated in a separate work, where Theorem 3.11 will be extended to any local ring A and \mathfrak{g}_A -module M of finite type over A . Local systems, however, are basic, since then the computation of the Hilbert quasi-polynomial is reduced to that of a module of invariants over a Lie subalgebra of the fibre Lie algebra, while the general case involves the hard problem of computing differential Galois groups for partial differential equations. Moreover, there are important situations when we know that M is a local system. One is when the maximal defining ideal is a maximal ideal in the ordinary sense, another when A is regular and is either \mathfrak{m}_A -adically complete, or an analytic algebra over the complex numbers. Yet another favourable case is worked out in Section 4, where we exemplify the general results by studying ideals in an allowed regular ring A that are preserved by Lie algebroids containing sub-algebroids that map to a toral subalgebra of T_A . This gives an alternative approach to the study of monomial ideals in a regular local ring, emphasising its symmetries. Finally, if a \mathfrak{g}_A -module M is cyclic over A , then the \mathfrak{g}_R -module $G_{J_m}^\bullet(M)$ is a (direct sum of) local systems; this is proven for principal ideals in Proposition 4.1, but the general case is similar.

In Section 5 we study Hilbert series of complex analytic singularities. The general set-up is an ideal $I \subset \mathfrak{m}$, where \mathfrak{m} is the maximal ideal of the ring \mathcal{O}_n of convergent power series in n variables, and $\mathfrak{g} = T_{\mathcal{O}_n}(I)$, where we can assume also without loss of generality that $\mathfrak{g} \subset T_{\mathcal{O}_n}(\mathfrak{m})$. Let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}/\mathfrak{m}\mathfrak{g}$ be the fibre Lie algebra. The Hilbert series

$$H_M(t) = \sum_{i \geq 0} \ell_{\mathfrak{g}}\left(\frac{\mathfrak{m}^i M}{\mathfrak{m}^{i+1} M}\right) t^i = \sum_{i \geq 0} \ell_{\mathfrak{g}_{\mathbf{C}}}\left(\frac{\mathfrak{m}^i M}{\mathfrak{m}^{i+1} M}\right) t^i$$

is a useful summary of a \mathfrak{g} -module M of finite type over \mathcal{O}_n . The first question is to determine when the fibre Lie algebra $\mathfrak{g}_{\mathbf{C}}$ is solvable, as the Hilbert series will then coincide with the ordinary Hilbert series and thus give us a rather good control of the \mathfrak{g} -module structure. We prove that if \mathcal{O}_n/I is a complete intersection ring with an isolated singularity and $I \subset \mathfrak{m}^2$ ($I \subset \mathfrak{m}^3$ when I is principal), then $\mathfrak{g}_{\mathbf{C}}$ is solvable (Th. 5.3), which was proven for hypersurfaces by Granger and Schulze [7]. Now put $J = (f + T_{\mathcal{O}_n} \cdot f)$, where $f \in \mathfrak{m}$, and assume that $\mathcal{O}_n/(f)$ has an isolated singularity. We prove that the fibre Lie algebra of the Lie algebroid $T_{\mathcal{O}_n}(J)$ is solvable (Th. 5.5), where the proof is based on Schulze and Yau's result [28, 32] that the derivation algebra T_A of the modular algebra $A = \mathcal{O}_n/J$ is solvable. We remark also, by a theorem of Yau and Mather [18] that the Hilbert series $H_A(t) = \sum_{i \geq 0} \dim_{\mathbf{C}}(J^i/J^{i+1})t^i$ of

the isolated singularity $B = \mathcal{O}_n/(f)$ is completely determined by the algebraic structure of the degree zero part A (see Theorem 5.7), so there arises a natural converse problem: If two hypersurfaces B and $B' = \mathcal{O}_n/(f')$, with isolated singularities and equal dimension, have equal Hilbert series $H_A(t) = H_{A'}(t)$, how are the rings B and B' then related?

In Section 2 we work out some basic results for modules M over Lie algebroids \mathfrak{g}_A , which are of finite type over A . First we give some salient relations between the length of A and M , where we want to emphasise the importance of Proposition 2.10, (4), for the very definition of Hilbert series (*Prop.* 2.21). Then the notion of local system is explained, and we give some examples; in particular we show how representations of Lie algebras give rise to local systems after localisation. In Section 2.4.1 we describe a stratification of the set of closed points in a schemes X of finite type over k (more general schemes are also allowed) using the notion of *defining point*, which is a preserved point x such that the tangent sheaf acts transitively on the residue field: $T_{X,x} = T_{X,x}(\mathfrak{m}_{X,x})$ and $T_{X,x} \rightarrow T_{k_{X,x}/k}$ is surjective. This gives in particular an algebraic version of the logarithmic stratification of a hypersurface singularity, introduced in the analytic case by Saito [25], and more generally, replacing the tangent sheaf by an arbitrary sheaf of Lie subalgebroids of the tangent sheaf, $\mathfrak{g}_X \subset T_X$, we arrive at an algebraic version of the Frobenius theorem for involutive systems of vector fields. Hilbert's theorem of finite generation of invariant rings and modules is generalised to algebras over Lie algebroids (*Th.* 2.25). We also describe a construction of noetherian graded \mathfrak{g}_k -subalgebras of noetherian graded \mathfrak{g}_k -algebras when \mathfrak{g}_k is solvable (*Th.* 2.28), which we use to refine our results about Hilbert series by selecting subsets of highest weights satisfying certain conditions. To determine the structure of fibre Lie algebras of Lie algebroids that contain a (weakly) toral subalgebra we need a recognition theorem (*Th.* 3.17), which determines the structure of the semi-simple part of a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_k(V)$ that contains a Cartan subalgebra of $\mathfrak{sl}_k(V)$.

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2. Lie algebroids, modules and defining ideals

2.1. Allowed rings

Let k be a field that contains the rational numbers \mathbf{Q} . We will deal with noetherian commutative k -algebras A/k such that the A -module of k -linear derivations $T_A := T_{A/k}$ is “big enough”, so that in particular the Jacobian criterion of regularity applies.

First we recall that any local ring A containing the rational numbers also contains a subfield $k \subset A$ such that $k \rightarrow k_A = A/\mathfrak{m}_A$ is 0-etale, so in particular $T_{k_A/k} = 0$.

Theorem 2.1. (*[20, Thms. 30.6, 30.8]*) *Let (R, \mathfrak{m}_R) be a regular local ring of dimension n containing the rational numbers \mathbf{Q} . Let k be a quasi-coefficient field of R and K be a coefficient field of its completion R^* such that $k \subset K$. The following conditions are equivalent:*

- (1) *There exist $\partial_1, \dots, \partial_n \in T_R$ and $f_1, \dots, f_n \in \mathfrak{m}_R$ such that $\det \partial_i(f_j) \notin \mathfrak{m}_R$.*

(2) If $\{x_1, \dots, x_n\}$ is a regular system of parameters and ∂_{x_i} are the partial derivatives of $R^* = K[[x_1, \dots, x_n]]$, $\partial_{x_i}(x_j) = \delta_{ij}$, then $\partial_{x_i} \in T_R$.

(3) T_R is free of rank n .

Furthermore, if these conditions hold, then for any $P \in \text{Spec } R$, putting $A = R/P$, we have $T_A = T_R(P)/PT_R$, and $\text{rank } T_A = \dim A$.

If the equivalent conditions in Theorem 2.1 hold, then we say that (R, \mathfrak{m}_R) satisfies the weak Jacobian condition $(WJ)_k$. We say also that a local ring A is *allowed* if $A = R/I$ where R satisfies $(WJ)_k$ and I is an ideal of R . For example, A is allowed if it is a quotient of R when R is a localisation of a polynomial ring, a formal power series ring, or a ring of convergent power series when k is either the field of real or complex numbers. We always make the following assumption:

- All local rings A/k are allowed, and k is an algebraically closed field of characteristic 0.

For an ideal I of height r we let J be the ideal that is generated by all the determinants $\det(\partial_i(f_j))$, where $\partial_i \in T_R$ and $f_j \in I$, $1 \leq i, j \leq r$. It is straightforward to see that $T_R(I) \cdot J \subset J$ (see e.g. [14]). Therefore the *Jacobian ideal* $\bar{J} = AJ$ is a T_A -submodule of A .

Recall that the ring of differential operators $\mathcal{D}_A = \mathcal{D}_{A/k} \subset \text{End}_k(A)$ is defined inductively as $\mathcal{D}_A = \cup_{m \geq 0} \mathcal{D}_A^m$, $\mathcal{D}_A^0 = \text{End}_A(A) = A$, $\mathcal{D}_A^{m+1} = \{P \in \text{End}_k(A) \mid [P, A] \subset \mathcal{D}_A^m\}$, where $[P, A] = PA - AP \subset \text{End}_k(A)$. It is easy to see that $T_A \subset \mathcal{D}_A^1 \subset \mathcal{D}_A$, and conversely, if $P \in \mathcal{D}_A^1$, then $P - P(1) \in T_A$; hence

$$\mathcal{D}_A^1 = A + T_A.$$

In general the algebra \mathcal{D}_A need not be generated by the A -submodule \mathcal{D}_A^1 , as was first exemplified in [3]. On the other hand we have the following companion to Theorem 2.1, which should be well known although we were unable to find a really complete proof in the literature; the cases when R is essentially of finite type or the ring of convergent power series have been treated in [2, 11, 19, 31], but we want to stress that the result applies to any allowed regular local k -algebra of characteristic 0.

Proposition 2.2. *Let R/k be an allowed regular local k -algebra. Then \mathcal{D}_R^1 generates the algebra \mathcal{D}_R .*

Select x_i and ∂_{x_i} as in Theorem 2.1. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, $n = \dim R$, we put $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \in R$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n} \in \mathcal{D}_R$, $|\alpha| = \sum \alpha_i$, and $\alpha! = \alpha_1! \dots \alpha_n!$.

Lemma 2.3. *If $P \in \mathcal{D}_R^m$ and $P(X^\alpha) = 0$ when $|\alpha| \leq m$, then $P = 0$.*

Proof. We use induction over m . If $P \in \mathcal{D}_R^0 = R$, then $P = P(1) = 0$. Assume the assertion is true when $P \in \mathcal{D}_R^m$, and let now $P \in \mathcal{D}_R^{m+1}$, such that $P(X^\alpha) = 0$ when $|\alpha| \leq m+1$. Then $P^{(i)} = [P, x_i] \in \mathcal{D}_R^m$, and $P^{(i)}(X^\alpha) = P(x_i X^\alpha) - x_i P(X^\alpha) = 0$ when $|\alpha| \leq m$, so by induction, $P^{(i)} = 0$. Therefore $[P, X^\alpha] = 0$, and since $P(1) = 0$, we get $P(X^\alpha) = 0$ for any monomial X^α . Expanding an element $f \in R$ in the form $f = \sum_\alpha c_\alpha X^\alpha + f_{i+1}$, where $c_\alpha \in k$ and $f_{i+1} \in \mathfrak{m}_R^{i+1}$, it follows that $P(f) \in \cap_{i \geq 0} \mathfrak{m}_R^i = \{0\}$, by Krull's intersection theorem. Therefore $P = 0$. \square

Proof of Proposition 2.2. Let $\mathcal{D}(T_R) \subset \mathcal{D}_R$ be the subalgebra that is generated by \mathcal{D}_R^1 . If $P \in \mathcal{D}_R^m$, define inductively for α such that $|\alpha| \leq m$, $a_0 = P(1)$, and $a_\alpha = \frac{1}{\alpha!}(P(X^\alpha) - \sum_{|\beta| < |\alpha|} a_\beta \partial^\beta(X^\alpha))$, so $\sum_\alpha a_\alpha \partial^\alpha \in \mathcal{D}_R$. One checks that $P - \sum_\alpha a_\alpha \partial^\alpha$ kills all monomials X^α such that $|\alpha| \leq m$, hence by Lemma 2.3 $P = \sum_\alpha a_\alpha \partial^\alpha \in \mathcal{D}(T_R)$. \square

2.2. Modules over Lie algebroids

A Lie algebroid over A/k is an A -module \mathfrak{g}_A , always assumed to be of finite type, which is equipped with a structure of Lie algebra over k and a homomorphism of Lie algebras and A -modules $\alpha : \mathfrak{g}_A \rightarrow T_A$ such that $[\delta, r\eta] = \alpha(\delta)(r)\eta + r[\delta, \eta]$, $\delta, \eta \in \mathfrak{g}_A, r \in A$. For example, T_A is a Lie algebroid, with $\alpha = \text{id}$. Let M be an A -module, define the map $i : A \rightarrow \text{End}_k M$, $i(a)(m) = am$, and let $\mathcal{D}_A^1(M) = \{p \in \text{End}_k(M) \mid [p, \text{End}_A(M)] \subset \text{End}_A(M)\} = \{p \in \text{End}_k(M) \mid [p, i(A)] \subset \text{End}_A(M)\}$ be its module of first order differential operators, so $i(A) \subset \mathcal{D}_A^1(M)$. We have the A -submodule

$$\mathfrak{c}_A(M) = \{\delta \in \mathcal{D}_A^1(M) \mid [\delta, i(A)] \subset i(A)\} \subset \mathcal{D}_A^1(M),$$

which is moreover a Lie subalgebra over k of $\mathcal{D}_A^1(M)$; if $\text{Ann } M = 0$, so the map i is injective, there exists a natural map $\beta : \mathfrak{c}_A(M) \rightarrow T_A$, giving $\mathfrak{c}_A(M)$ a structure of Lie algebroid (see [14]); if $\delta \in \mathfrak{c}_A(M)$, then $\beta(\delta)$ is determined by the relation $[\delta, i(a)] = i(\beta(\delta)(a))$. More generally, we have the Lie algebroid $\beta : i(A) \otimes_A \mathfrak{c}_A(M) = \mathfrak{c}_{i(A)}(M) \rightarrow T_{i(A)}$, where M is considered as $i(A)$ -module. A module over a Lie algebroid \mathfrak{g}_A is an A -module M and a homomorphism of A -modules and Lie algebras $\rho : \mathfrak{g}_A \rightarrow \mathfrak{c}_A(M)$. In general the difference $\gamma = i(\alpha(\delta)(a)) - [\rho(\delta), i(a)] \in i(A)$, $a \in A, \delta \in \mathfrak{g}_A$, need not equal 0 (take $\rho = 0$ and M torsion free, for instance), but if $\text{Ann}_A(\rho(\mathfrak{g}_A)) = 0$ (a sufficient condition is that M is torsion free and $\rho \neq 0$), then $\gamma = 0$ and moreover $\beta \circ \rho(\delta) = i(\alpha(\delta))$. In general, if M is of finite length as \mathfrak{g}_A -module, then $\gamma \in \sqrt{\text{Ann } M}$. If S is a multiplicative system in A , then there is a natural homomorphism $\mathfrak{c}_A(M) \rightarrow \mathfrak{c}_A(MS^{-1})$, $\delta(m/s) = \delta(m)/s - \beta(\delta)(i(s))m/s^2$, so in particular, if P is a prime ideal, then there is a natural homomorphism of A -modules and Lie algebras $\mathfrak{c}_A(M) \rightarrow \mathfrak{c}_A(M_P)$.

For an ideal $I \subset A$ we have inclusions

$$T_A(I) \subset \mathcal{D}_{A,I}^1 = \{\delta \in \mathcal{D}_A^1 \mid \delta \cdot I \subset I\} = A + T_A(I) \subset \mathfrak{c}_A(I) \subset \mathcal{D}_A^1(I),$$

where $T_A(I) \subset \mathcal{D}_{A,I}^1$ are the derivations (differential operators of first order) of A that preserve I , while $\mathcal{D}_A^1(I)$ are the first order differential operators of I . The notation is slightly inconsistent since on the one hand $T_A(I) \subset T_A$, while on the other $\mathfrak{c}_A(I) \subset \mathcal{D}_A^1(I)$ are not subsets of \mathcal{D}_A^1 .

Assuming for simplicity that A is an integral domain, we have the exact sequence

$$0 \rightarrow \text{Hom}_A(I, I) \rightarrow \mathfrak{c}_A(I) \xrightarrow{\beta} T_A.$$

Proposition 2.4. *Assume that A is a normal integral domain, and $I = (x_1, \dots, x_r)$ is a non-zero ideal of A . Then $\text{Hom}_A(I, I) = A$, and we have:*

$$(1) \quad \beta^{-1}(T_A(I)) = A + T_A(I).$$

(2)

$$\text{Im}(\beta) = \{\partial \in T_A \mid \frac{\partial(x_i)}{x_i} - \frac{\partial(x_j)}{x_j} \in I/x_i + I/x_j, 1 \leq i, j \leq r\},$$

where $I/x_i + I/x_j \subset K(A)$ is the fractional ideal of A which is generated by the elements $x_k/x_i, x_k/x_j, 1 \leq k \leq r$.

(3) If $\{x_1, \dots, x_r\}$ is a regular sequence of length $r \geq 2$, then

$$\mathfrak{c}_A(I) = A + T_A(I).$$

Proof. If $I = A$, then all assertions are easily seen to be true, so assume $I \neq A$. We have $\mathfrak{c}_A(I) \subset \mathfrak{c}_A(K) = K + T_K$, where $K = K(A)$ is the fraction field of A . Moreover, if $\delta \in \mathfrak{c}_A(I)$, then $\beta(\delta) = \partial \in T_A$ and $\delta = r + \partial$, where $r \in K$. We first show for completeness that $\text{Hom}_A(I, I) = A$: If $r = a/b \in K \cap \text{Hom}_A(I, I)$, where $a, b \in A$, and $rI \subset I$, we have $aI \subset bI$. If on the contrary b does not divide a , as A is the intersection of all its 1-dimensional localisations, there exists a prime ideal P of height 1 such that $b \in P$ but $a \notin P$, so that localising at P one concludes that $I \subset PI$, which by Nakayama's lemma would imply that $I = 0$ since A is an integral domain. Therefore b divides a .

(1): If $\partial \in T_A(I)$ and $x \in I$, then $rx + \partial(x) \in I$, where $\partial(x) \in I$, so $rx \in I$; hence $r \in \text{Hom}_A(I, I) = A$.

(2): Since $rx + \partial(x) \in I$ for all $x \in I$, then $r = -\partial(x)/x + s/x$ for some $s \in I$. Therefore $\partial(x)/x - \partial(x')/x' \in I/x + I/x'$ for all $x, x' \in I$. Conversely, if $\partial(x_i)/x_i - \partial(x_j)/x_j = s_i/x_i + s_j/x_j, 1 \leq i, j \leq r, s_i, s_j \in I$, then $-\partial(x_i)/x_i + s_i/x_i + \partial = -\partial(x_j)/x_j + s_j/x_j + \partial \in \mathfrak{c}_A(I)$.

(3): In (2) we have $x_j(\partial(x_i) + s_1) = x_i(\partial(x_j) + s_2)$ for some $s_1, s_2 \in I$. Since A is a local ring, any permutation of the regular sequence is again a regular sequence, hence $\{x_i, x_j\}$ is A -regular. Therefore $\partial(x_i) + s_1 \in (x_i)$, so $\partial(x_i) \in I$. This implies $\partial \in T_A(I)$. \square

Example 2.5. If $I = (x) \in A = k[x, y]$, then $\mathfrak{c}_A(I) = A + A(-1/x + \partial_x) + A\partial_y$, and $\text{Im}(\beta) = T_A$. If $I = (x, y)$, then $\mathfrak{c}_A(I) = A + (x, y)T_A$.

By Proposition 2.2, T_R -modules are the same as modules over the ring of differential operators \mathcal{D}_R when R is a regular allowed local k -algebra over a field of characteristic 0. Since A is noetherian, if M is a \mathfrak{g}_A -module of finite type over A , then M is noetherian as \mathfrak{g}_A -module and contains in particular a maximal proper \mathfrak{g}_A -submodule $N \subset M$, so the quotient M/N is a simple \mathfrak{g}_A -module.

We say that \mathfrak{g}_A acts on an A -module M if we are given a map (not necessarily A -linear) $\rho : \mathfrak{g}_A \rightarrow \mathfrak{c}_A(M)$ such that $\rho(\delta)(rm) = \alpha(\delta)(r)m + r\rho(\delta)(m), \delta \in \mathfrak{g}_A, r \in A, m \in M$. For example, if M is a \mathfrak{g}_A -module then \mathfrak{g}_A acts on M , while T_A acts on itself, but T_A is not a T_A -module.

Proposition 2.6. Let \mathfrak{g}_A be a Lie algebroid such that A is simple. If \mathfrak{g}_A acts on an A -module of finite type M , then M is free. In particular, \mathfrak{g}_A is free over A .

Proof. The Fitting ideals of M are \mathfrak{g}_A -invariant ideals of A (see e.g. [14]). Since A is simple it follows that all Fitting ideals equals either 0 or A , implying M is free. A Lie algebroid \mathfrak{g}_A acts on itself by the adjoint action $\rho(\delta)(\eta) = [\delta, \eta]$, therefore \mathfrak{g}_A is free. \square

Proposition 2.7. *Let A be an allowed local ring. Consider the conditions:*

- (1) A is regular.
- (2) $\alpha : \mathfrak{g}_A \rightarrow T_A$ is surjective (we say that \mathfrak{g}_A is transitive).
- (3) A is a simple \mathfrak{g}_A -module.
- (4) The depths $\text{depth } A \geq 2$, $\text{depth } \alpha(\mathfrak{g}_A) \geq 2$, and $\mathfrak{m}_A^d T_A \subset \alpha(\mathfrak{g}_A)$ for some integer $d \geq 0$.

We have (3) \Rightarrow (1), (1 – 2) \Rightarrow (3), and (4) \Rightarrow (2).

Assume $l_{\mathfrak{g}_A}(A) < \infty$. If $I \subset \mathfrak{m}_A$ is an invariant ideal, $\mathfrak{g}_A \cdot I \subset I$, then I belongs to the nilradical $\text{nil } A$. The reduced ring $A/\text{nil } A$ is regular.

Remark 2.8. (1) (3) in Proposition 2.7 does not imply (2). Derivations of a regular noetherian ring which have no proper invariant ideals are studied, for example, in [10, 13].

- (2) (3) implies (1), hence since A is allowed, T_A is free. By Proposition 2.6 $\alpha(\mathfrak{g}_A)$ is free, and if $\dim A \geq 2$, then $\alpha(\mathfrak{g}_A) \subset T_A$ is an inclusion of modules of depth ≥ 2 . If A is simple over \mathfrak{g}_A , hence regular, and $\mathfrak{m}_A^d T_A \subset \alpha(\mathfrak{g}_A)$, one can prove directly in local coordinates that (2) follows.

Proof. (3) \Rightarrow (1): The Jacobian ideal \bar{J} is a non-zero \mathfrak{g}_A -submodule of the simple \mathfrak{g}_A -module A , hence $\bar{J} = A$. Since A is allowed, it follows that A is regular (Th. 2.1).

(1 – 2) \Rightarrow (3): Since A is regular and T_A is free Theorem 2.1 implies that A has a regular system of parameters x_1, \dots, x_r , $r = \dim A$, and derivations $\partial_1, \dots, \partial_r \in T_A$ satisfying $\partial_i(x_j) = \delta_{ij}$. Let I be a T_A -invariant non-zero ideal of A and $f \in I$ and $f \in \mathfrak{m}_A^l$ with minimal l . If $l \geq 1$ there exists a derivation $\delta = \sum a_i \partial_i \in T_A$ such that $\delta(f) \in \mathfrak{m}_A^{l-1} \setminus \mathfrak{m}_A^l$. It follows that $I = A$. Since α is surjective this implies that A is simple over \mathfrak{g}_A .

(4) \Rightarrow (2): By assumption the map of stalks $(\alpha(\mathfrak{g}_A))_q = (T_A)_q$ when q is a non-closed point in $\text{Spec } A$. Hence if $\delta \in T_A$, then $\delta_q \in (\alpha(\mathfrak{g}_A))_q$ for such points q . Since $\text{depth } \alpha(\mathfrak{g}_A) \geq 2$ it follows that $\delta \in \alpha(\mathfrak{g}_A)$.

We prove the last assertion. Since $\text{Char } k = 0$ the nilradical $\text{nil } A$ of A is preserved by \mathfrak{g}_A (see e.g. [14]), so $A' = A/\text{nil } A$ is a \mathfrak{g}_A -module of finite length. If, on the contrary, $\bar{I} = I \bmod \text{nil } A$ is a non-zero ideal in $\mathfrak{m}_{A'}$, by Nakayama's lemma and since A' is reduced, $\bar{I}^n \subset \mathfrak{m}_{A'}$, $n = 1, 2, \dots$ is a strictly descending sequence of \mathfrak{g}_A -invariant ideals, so A' will not have a finite length. Therefore $I \subset \text{nil } A$. It follows that $\text{nil } A$ is a maximal \mathfrak{g}_A -submodule of A , hence $A/\text{nil } A$ is simple, and therefore it is regular, since (3) implies (1). \square

We have avoided the language of sheaves but make this one exception, assuming it is known what a sheaf of Lie algebroids is, and the corresponding notion of sheaf of modules. Thus if M is a \mathfrak{g}_A -module and $U \subset \text{Spec } A$ is an open set, we have the sheaf M_U of \mathfrak{g}_U -modules. We say that a sheaf of \mathfrak{g}_U -modules is simple if it contains no proper quasi-coherent sheaf of submodules.

Proposition 2.9. *Let L be a simple \mathfrak{g}_A -module and $U \subset \text{Spec } A$ be a non-empty connected open subset such that $U \cap \text{supp } L \neq \emptyset$. Then the associated sheaf L_U of \mathfrak{g}_U -modules is simple, and if $P \in \text{supp } L$, then the stalk L_P is simple over the Lie algebroid $A_P \otimes_A \mathfrak{g}_A$.*

Proof. It suffices to prove that the stalk L_P is simple when $P \in \text{supp } L$. Since the fundamental open sets $X \setminus V(f)$, $f \in A$, form a basis for the topology of $\text{Spec } A$ and L_U is coherent, it suffices to prove that the module $L_f = A_f \otimes_A L$ is simple when $\text{supp } L \not\subset V(f)$. To see this, let $N \subset L_f$ be a non-zero submodule, which is generated by elements l_i/f^{n_i} , $l_i \in L$. Then each nonzero l_i generates L over \mathfrak{g}_A , but this then implies that l_i/f^{n_i} generates L_f , and therefore $N = L_f$. \square

Proposition 2.10. *Let \mathfrak{g}_A be a Lie algebroid over a local k -algebra A , and M be \mathfrak{g}_A -module of finite type as A -module.*

- (1) *If M is simple and $\text{Ann } M = 0$, then M is free over A .*
- (2) *If A is simple over \mathfrak{g}_A , then $\ell_{\mathfrak{g}_A}(M) \leq \text{rank}(M)$.*
- (3) *If $\ell_{\mathfrak{g}_A}(A) < \infty$, then $\ell_{\mathfrak{g}_A}(M) < \infty$.*
- (4) *$\ell_{\mathfrak{g}_A}(A/\text{Ann } M) \leq \ell_{\mathfrak{g}_A}(M)$.*

Proof. (1): Let J be the first non-vanishing Fitting ideal of M . Since $\text{Ann } M = 0$ it follows that JM is a non-zero \mathfrak{g}_A -submodule of M , so $M = JM$. By Nakayama's lemma it follows that $J = A$, hence M is free.

(2): If $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ is a short exact sequence of \mathfrak{g}_A -modules, by Proposition 2.6 all these modules are free, and $\text{rank } M = \text{rank } M_1 + \text{rank } M_2$. From this follows that in any descending chain of \mathfrak{g}_A -modules each subquotient is free over A , and hence it is stationary, and that the number of simple subquotients cannot exceed $\text{rank } M$.

(3): Since $\ell_{\mathfrak{g}_A}(A) < \infty$ it follows from Proposition 2.7 that the maximal proper \mathfrak{g}_A -submodule I of A equals the nilradical. Put $B = A/I$ and $\mathfrak{g}_B = B \otimes_A \mathfrak{g}_A$, which is a Lie algebroid over the simple \mathfrak{g}_B -module B . Since $I^n = 0$ for high n , there exists a finite set Ω of indices, so that $i \in \Omega$ if and only if $I^i M / I^{i+1} M \neq 0$. Therefore by (2)

$$\ell_{\mathfrak{g}_A}(M) = \ell_{\mathfrak{g}_A}(\oplus_{i \in \Omega} \frac{I^i M}{I^{i+1} M}) = \sum_{i \in \Omega} \ell_{\mathfrak{g}_B}(\frac{I^i M}{I^{i+1} M}) < \infty.$$

(4): We use induction over $n = \ell_{\mathfrak{g}_A}(M)$. Since $\mathfrak{g}_A \cdot \text{Ann } M \subset \text{Ann } M$, putting $A_1 = A/\text{Ann } M$, then $\mathfrak{g}_{A_1} = \mathfrak{g}_A/\text{Ann } M \mathfrak{g}_A$ is a Lie algebroid over A_1 and M is a \mathfrak{g}_{A_1} -module such that $\text{Ann}_{A_1} M = 0$. Since $\ell_{\mathfrak{g}_{A_1}}(A_1) = \ell_{\mathfrak{g}_A}(A_1)$ and $\ell_{\mathfrak{g}_A}(M) = \ell_{\mathfrak{g}_{A_1}}(M)$, it suffices thus to prove

$$\ell_{\mathfrak{g}_{A_1}}(A_1) \leq \ell_{\mathfrak{g}_{A_1}}(M).$$

If M is simple and $I \subset A_1$ is a non-zero \mathfrak{g}_{A_1} -invariant ideal, then since $\text{Ann}_{A_1}(M) = 0$, IM is a non-zero \mathfrak{g}_{A_1} -submodule, hence $IM = M$ since M is simple, hence $I = A_1$ by Nakayama's lemma; hence A_1 is simple. This proves the assertion when $n = 1$. Let $J = \sqrt{(0)}$ be the nilradical of A_1 , $B = A_1/J$, and $G(M) = \oplus_{i \geq 0} J^i M / J^{i+1} M$. Put also $\mathfrak{g}_B = \mathfrak{g}_{A_1}/J \mathfrak{g}_{A_1}$, which is a Lie algebroid over B since $\mathfrak{g}_{A_1} \cdot J \subset J$ (see [14, 26]). Then $G(M)$ and $G(A_1)$ are \mathfrak{g}_B -modules, and $\text{Ann}_B(G(M)) = 0$. Moreover, $\ell_{\mathfrak{g}_{A_1}}(A_1) = \ell_{\mathfrak{g}_B}(G(A_1))$ and $\ell_{\mathfrak{g}_{A_1}}(M) = \ell_{\mathfrak{g}_B}(G(M))$, so it is equivalent to prove

$$\ell_{\mathfrak{g}_B}(G(A_1)) \leq \ell_{\mathfrak{g}_B}(G(M))$$

when $\text{Ann}_B G(M) = \{0\}$, which we thus know is true when M is simple. Assume $n > 1$ and that the assertion is true for all \mathfrak{g}_A and M such that $\ell_{\mathfrak{g}_A}(M) \leq n - 1$; hence the above inequality holds when $\ell_{\mathfrak{g}_B}(G(M)) = \ell_{\mathfrak{g}_A}(M) \leq n - 1$. If there exists a proper submodule $M_1 \subset M$ such that $\text{Ann}_{A_1}(M_1) = 0$, then by induction $\ell_{\mathfrak{g}_{A_1}}(M) > \ell_{\mathfrak{g}_{A_1}}(M_1) \geq \ell_{\mathfrak{g}_{A_1}}(A_1)$. If no such submodule exists, then since $\ell_{\mathfrak{g}_{A_1}}(M) \geq 2$ there exists a non-zero \mathfrak{g}_A -submodule $M_1 \subset M$ such that $\text{supp } M = \text{supp } M/M_1$. We have then a surjective homomorphism of \mathfrak{g}_B -modules $G(M) \rightarrow G(M/M_1)$, where $\text{supp } G(M/M_1) = \text{supp } G(M) = \text{Spec } B$; hence since B is a reduced ring, $\text{Ann}_B G(M/M_1) = \text{Ann}_B G(M) = \{0\}$. Therefore, by induction, $\ell_{\mathfrak{g}_B}(G(A_1)) \leq \ell_{\mathfrak{g}_B}(G(M/M_1)) < \ell_{\mathfrak{g}_B}(G(M))$. \square

Say that a Lie algebroid is split, or has an integrable connection, if the exact sequence (2.1) has a split $\phi : \bar{\mathfrak{g}}_A \rightarrow \mathfrak{g}_A$ as A -modules and Lie algebras. If a split ϕ is chosen, we write $\mathfrak{g}_A = \mathfrak{g} \rtimes \bar{\mathfrak{g}}_A$ and call it a semi-direct product of $\bar{\mathfrak{g}}_A$ by \mathfrak{g} . Transitive Lie algebroids over rings that are simple over T_R (e.g. R is regular and allowed) have the form $\mathfrak{g} \oplus T_R$ as R -module, but need not be isomorphic to $\mathfrak{g} \rtimes T_R$, so any choice of split $T_R \rightarrow \mathfrak{g}_R$ of α need not be a homomorphism of Lie algebroids (so all connections are non-integrable). However, if we are studying \mathfrak{g}_R -modules M of finite type over R , so M is free over R (Prop. 2.6), then its associated Lie algebroid $\mathfrak{c}_R(M)$ has an integrable connection, although such a connection need not arise from a connection of \mathfrak{g}_R . More precisely, given the integrable connection $\nabla : T_R \rightarrow \mathfrak{c}_{R/k}(M)$ there need not exist a map $\hat{\nabla} : T_R \rightarrow \mathfrak{g}_R$ such that $\nabla = \rho \circ \hat{\nabla}$, where $\rho : \mathfrak{g}_R \rightarrow \mathfrak{c}_R(M)$ is the action of \mathfrak{g}_R on M .

2.2.1. Forgetting the A -module structure

Forgetting the A -module structure gives a functor $\mathfrak{g}_A \mapsto \mathfrak{g}_A^{Lie}$ from the category of Lie algebroids to the category of Lie algebras. There is an associated functor $\text{Mod}(\mathfrak{g}_A, A) \rightarrow \text{Mod}(\mathfrak{g}_A^{Lie})$, $M \mapsto L(M)$, where $L(M)$ is M regarded as \mathfrak{g}_A^{Lie} -module. Since modules over the Lie algebra \mathfrak{g}_A^{Lie} in general need not be modules over the Lie algebroid \mathfrak{g}_A we only have an inequality of lengths

$$\ell_{\mathfrak{g}_A}(M) \leq \ell_{\mathfrak{g}_A^{Lie}}(L(M)),$$

but there are situations when we have some control of the difference. When R is simple over \mathfrak{g}_R and M is a \mathfrak{g}_R -module of finite type over R , we let $\ell_{\mathfrak{g}_R}^{inv}(M)$ be the multiplicity of R in M , i.e. the number of simple subquotients in a composition series of M that are isomorphic to R . For a \mathfrak{g}_A^{Lie} -module, denote as usual $N^{\mathfrak{g}_A} = \{n \in N \mid \mathfrak{g}_A \cdot n = 0\}$, and say that N has no \mathfrak{g}_A -invariants if $N^{\mathfrak{g}_A} = 0$.

Proposition 2.11. *Let A be an allowed local ring and \mathfrak{g}_A be a Lie algebroid over A .*

- (1) *Let M be a \mathfrak{g}_A -module such that $\ell_{\mathfrak{g}_A^{Lie}}(L(M)) < \infty$ and such that each simple subquotient of $L(M)$ contains no \mathfrak{g}_A -invariants. Then $\ell_{\mathfrak{g}_A^{Lie}}(L(M)) = \ell_{\mathfrak{g}_A}(M)$.*
- (2) *Assume that R is simple over the Lie algebroid \mathfrak{g}_R and let M be a \mathfrak{g}_R -module of finite type. Then $\ell_{\mathfrak{g}_R^{Lie}}(L(M)) = \ell_{\mathfrak{g}_R}(M) + \ell_{\mathfrak{g}_R}^{inv}(M)$.*

Proof. (1): We use induction over $\ell_{\mathfrak{g}_A^{Lie}}(L(M))$. If $L(M)$ is simple as \mathfrak{g}_A^{Lie} -module, then clearly M is simple. Assume $\ell_{\mathfrak{g}_A^{Lie}}(L(M)) = \ell_{\mathfrak{g}_A}(M)$ for all \mathfrak{g}_A -modules as in (1) such that $\ell_{\mathfrak{g}_A^{Lie}}(L(M)) \leq \ell$. If $\ell_{\mathfrak{g}_A^{Lie}}(L(M)) = \ell + 1$ we let $N \subset L(M)$ be a simple \mathfrak{g}_A^{Lie} -submodule, so by assumption $N^{\mathfrak{g}_A} = 0$ and therefore $N = \mathfrak{g}_A \cdot N$. Take $a \in A$ and $n \in N$, so $n = \delta \cdot n_1$ for some $n_1 \in N$, hence $a \cdot n = (a\delta) \cdot n_1 \in N$, since N is a subset of a \mathfrak{g}_A -module. Hence N is a simple \mathfrak{g}_A -submodule, and M/N is a \mathfrak{g}_A -module such that $\ell_{\mathfrak{g}_A^{Lie}}(L(M/N)) = \ell$, so by induction $\ell_{\mathfrak{g}_A}(M/N) = \ell$, and therefore $\ell_{\mathfrak{g}_A}(M) = \ell + 1 = \ell_{\mathfrak{g}_A^{Lie}}(L(M))$.

(2): We first prove $\ell_{\mathfrak{g}_R^{Lie}}(L(R)) = 2$. Since R is simple over \mathfrak{g}_R it follows that $R^{\mathfrak{g}_R} = k$, hence $R^{\mathfrak{g}_R}$ is a simple \mathfrak{g}_R^{Lie} -module. It remains to see that $\bar{R} = L(R)/k$ is simple over \mathfrak{g}_R^{Lie} . If $N \subset L(R)/k$ is a non-zero \mathfrak{g}_R^{Lie} -submodule, put $\hat{N} = p^{-1}(N) \subset L(R)$, where $p : L(R) \rightarrow L(R)/k$ is the natural projection. Then \hat{N} is a \mathfrak{g}_R^{Lie} -submodule such that $\hat{N}^{\mathfrak{g}_R} = k$, and $\mathfrak{g}_R \cdot \hat{N} \neq 0$, so $\mathfrak{g}_R \cdot \hat{N}$ is a non-zero \mathfrak{g}_R -submodule of \hat{N} and R , hence since R is simple, $\mathfrak{g}_R \cdot \hat{N} = R$. Therefore $\hat{N} = R$. Therefore $N = p(\hat{N}) = \bar{R}$.

By Proposition 2.10, $\ell_{\mathfrak{g}_R}(M) < \infty$. It suffices therefore to prove the following assertion: If $\ell_{\mathfrak{g}_R}(M) = 1$, then $\ell_{\mathfrak{g}_R^{Lie}}(L(M)) = 1$ when $M^{\mathfrak{g}_R} = 0$, and $\ell_{\mathfrak{g}_R^{Lie}}(L(M)) = 2$ when $M^{\mathfrak{g}_R} \neq 0$. Let $N \subset M$ be a non-zero \mathfrak{g}_R^{Lie} -submodule, hence $M = RN$, since M is simple. If $\delta \cdot n \in \mathfrak{g}_R \cdot N$ and $a \in R$, then $a \cdot (\delta \cdot n) = (a\delta) \cdot n \in \mathfrak{g}_R N$, so $\mathfrak{g}_R \cdot N$ is a \mathfrak{g}_R -submodule; hence $\mathfrak{g}_R N \subset N \subset RN = M$, so $N = M$, if $M^{\mathfrak{g}_R} = 0$, since M is simple. If $M^{\mathfrak{g}_R} \neq 0$, by simplicity $M \cong R$, hence $\ell_{\mathfrak{g}_R^{Lie}}(L(M)) = 2$. \square

2.3. Local systems

Let (R, \mathfrak{m}_R) be a local noetherian k -algebra which is simple over a Lie algebroid \mathfrak{g}_R , so we have the exact sequence

$$(2.1) \quad 0 \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}_R \rightarrow \bar{\mathfrak{g}}_R \rightarrow 0,$$

where \mathfrak{g} is a Lie algebra over R and $\bar{\mathfrak{g}}_R$ is a Lie subalgebroid of T_R . Put $\mathfrak{g}_k = k \otimes_R \mathfrak{g}$, which is a finite-dimensional Lie algebra over k , which we call the *fibre Lie algebra* of \mathfrak{g}_R . We have a natural specialisation homomorphism of Lie algebras $f : \mathfrak{g} \rightarrow \mathfrak{g}_k$, $\delta \mapsto 1 \otimes \delta$.

Since R is simple, so \mathfrak{g} and M are free over R (Prop. 2.6), any choice of basis of M and \mathfrak{g} induces an isomorphism of R -modules

$$\begin{aligned} M &\rightarrow R \otimes_k k \otimes_R M, \\ \mathfrak{g} &\rightarrow R \otimes_k \mathfrak{g}_k. \end{aligned}$$

However, the second map is in general not a homomorphism of Lie algebras over R , and the first is not a homomorphism of \mathfrak{g} -modules where \mathfrak{g} acts trivially on R and by the specialisation map f on $k \otimes_R M$, and even less a homomorphism of \mathfrak{g}_R -modules. Since M is free over R (Prop. 2.10) we nevertheless have an inequality of lengths

$$l_{\mathfrak{g}_R}(M) \leq l_{\mathfrak{g}_k}(k \otimes_R M).$$

Definition 2.12. Let \mathfrak{g}_R be a Lie algebroid such that R is simple over \mathfrak{g}_R . A *local system* over \mathfrak{g}_R is a \mathfrak{g}_R -module M which is of finite type as R -module and

$$l_{\mathfrak{g}_R}(M) = l_{\mathfrak{g}_k}(k \otimes_R M).$$

Let $Loc(\mathfrak{g}_R)$ be the category of local systems over \mathfrak{g}_R .

Since all \mathfrak{g}_R -subquotients of M are free over R (Prop. 2.6) it is straightforward to see that $M \in \text{Loc}(\mathfrak{g}_R)$ if and only if each simple subquotient of M induces a simple \mathfrak{g}_k -subquotient of $k \otimes_R M$.

Proposition 2.13. *Assume that R is simple over the Lie algebroid \mathfrak{g}_R and let*

$$0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$$

be an exact sequence of \mathfrak{g}_R -modules of finite type over R . Then $M \in \text{Loc}(\mathfrak{g}_R)$ if and only if $M_1, M_2 \in \text{Loc}(\mathfrak{g}_R)$.

Proof. Since M, M_1, M_2 are free over R , hence flat, it follows that

$$\ell_{\mathfrak{g}_R}(M) \leq \ell_{\mathfrak{g}_k}(k \otimes_R M) = \ell_{\mathfrak{g}_k}(k \otimes_R M_1) + \ell_{\mathfrak{g}_k}(k \otimes_R M_2) \geq \ell_{\mathfrak{g}_R}(M_1) + \ell_{\mathfrak{g}_R}(M_2).$$

Since $\ell_{\mathfrak{g}_R}(M) = \ell_{\mathfrak{g}_R}(M_1) + \ell_{\mathfrak{g}_R}(M_2)$, if one of the inequalities is an equality, then the other is too. Since moreover $\ell_{\mathfrak{g}_k}(k \otimes_R M_i) \geq \ell_{\mathfrak{g}_R}(M_i)$, $i = 1, 2$, the assertion follows. \square

Obviously, T_R -modules which of rank 1 over R are local systems, and we remark that in general there exist a great many non-isomorphic local systems of rank 1.

Example 2.14. Let $R = \mathbf{C}[x]$, so $T_R = R\partial_x$. The T_R -module $M_p = Re^p$, $p \in R$, is a simple T_R -module (Prop. 2.10). We have $M_p \cong M_q$ if and only if there exists $\phi \in R$ such that $(\partial_x - \partial_x(p))\phi e^q = 0$, i.e. $\partial_x(\phi)/\phi = \partial_x(p - q)$. Such ϕ exists if and only if $p - q \in \mathbf{C}$. Replacing R by the ring of convergent power series \mathcal{O}_1 , then all modules M_p are isomorphic.

Note that simple T_R -modules usually have rank > 1 as R -module and are then not local systems, and in general it is difficult to see when a given \mathfrak{g}_R -module is a local system. Still, we have some general conditions ensuring that a module be a local system. Before we state the first such condition, we recall that if $\{m_1, \dots, m_n\}$ is a basis of a \mathfrak{g}_R -module M , and $\{\delta_1, \dots, \delta_r\}$ is a set of generators of the Lie algebroid \mathfrak{g}_R (so they need not be generators as R -module), then the action of \mathfrak{g}_R on M is specified by the connection matrices $\Gamma(\delta_l) \in \text{Mat}_{n \times n}(R)$, $\delta_l \cdot m_i = \sum_j \Gamma(\delta_l)_{ij} m_j$.

Proposition 2.15. *Let (R, \mathfrak{g}_R) be a Lie algebroid where R is simple over $\bar{\mathfrak{g}}_R$, and M be a \mathfrak{g}_R -module of finite type over R . Assume that M has a basis $\{m_1, \dots, m_t\}$ and there exist elements $\{\delta_1, \dots, \delta_s\} \subset \mathfrak{g}_R$ whose image $\{\alpha(\delta_1), \dots, \alpha(\delta_s)\}$ generates $\bar{\mathfrak{g}}_R \subset T_R$, such that the connection matrices*

$$\Gamma(\delta_l) \in RI_n + \mathfrak{m}_R \text{Mat}_{n \times n}(R), \quad l = 1, \dots, s,$$

where I_n is the identity matrix. Then $M \in \text{Loc}(R)$.

Proof. Since $\ell_{\mathfrak{g}_R}(\cdot)$ and $\ell_{\mathfrak{g}_k}(k \otimes_R \cdot)$ are additive with respect to short exact sequences, remembering that all \mathfrak{g}_R -modules are free over R , it suffices to prove that if M is simple, then $k \otimes_R M$ is simple over \mathfrak{g}_k . Let \mathfrak{g}'_R be the Lie subalgebroid of \mathfrak{g}_R that is generated by the δ_i , so $\mathfrak{g}_R = \mathfrak{g} + \mathfrak{g}'_R$. Let $1 \otimes m \in k \otimes_R M$ be a non-zero element. We have $\delta_i \cdot m = \lambda m + m'$ where $m' \in \mathfrak{m}_R M$, $\lambda \in k$, hence for any $P \in \mathcal{D}(\mathfrak{g}'_R) \subset \mathcal{D}(\mathfrak{g}_R)$ we have $Pm \equiv \mu m \pmod{\mathfrak{m}_R M}$, $\mu \in k$. This implies

$$\mathcal{D}(\mathfrak{g}_k)(1 \otimes m) = 1 \otimes \mathcal{D}(\mathfrak{g})m = 1 \otimes \mathcal{D}(\mathfrak{g})\mathcal{D}(\mathfrak{g}'_R)m = 1 \otimes \mathcal{D}(\mathfrak{g}_R)m = 1 \otimes M.$$

\square

Example 2.16. Let $R = k[x]$ be the polynomial ring of one variable over k . The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(k)$ acts on R by the map $\alpha : \mathfrak{g} \rightarrow T_R$, $\alpha(H) = 2x\partial_x$, $\alpha(X_+) = x^2\partial_x$ and $\alpha(X_-) = -\partial_x$. Then $\mathfrak{g}_R = R \otimes_k \mathfrak{sl}_2(k)$ is a transitive Lie algebroid over R , $\mathfrak{b} := \text{Ker}(\mathfrak{g}_R \rightarrow T_R) = R(x \otimes H - 2 \otimes X_+) + R(1 \otimes H + 2x \otimes X_-)$, and we have the integrable connection $T_R \rightarrow \mathfrak{g}_R$, $a\partial_x \mapsto a \otimes X_-$, so $\mathfrak{g}_R = \mathfrak{b} \rtimes T_R$. The fibre Lie algebra $\mathfrak{g}_k = k \otimes_R \mathfrak{b}$ is a 2-dimensional solvable Lie algebra. If V is a finite-dimensional $\mathfrak{sl}_2(k)$ -module, we have the \mathfrak{g}_R -module $M = R \otimes_k V$, with action $r \otimes \delta \cdot (r' \otimes v) = r\alpha(\delta)(r') \otimes v + rr' \otimes \delta \cdot v$. We claim that

$$\ell_{\mathfrak{g}_R}(M) = \dim_k V = \ell_{\mathfrak{g}_k}(V),$$

and each simple subquotient of M is of rank 1 over R . Thus M is a local system over \mathfrak{g}_R . Proof: We can assume that V is simple. The Cartan algebra $\mathfrak{h} = kH$ gives a weight decomposition $M = \bigoplus_{\lambda \in \mathbf{Z}} M_\lambda$, where $Hm = \lambda m$ when $m \in M_\lambda$. Let λ_0 be the lowest integer such that $M_{\lambda_0} \neq 0$, so $\dim_k M_{\lambda_0+2i} = 1$ when $i = 0, 1, \dots, -\lambda_0$, and $\dim_k V = -\lambda_0 + 1$. We have $\dim M_\lambda - \dim M_{\lambda-2} = 1$ when $\lambda = \lambda_0 + 2i$, $i = 1, 2, \dots, -\lambda_0$, and $\dim M_\lambda = \dim M_{-\lambda_0}$ when $\lambda = -\lambda_0 + 2i$, $i = 0, \dots$. It follows that there exists a nonzero vector $m_{\lambda_0+2i} \in M_{\lambda_0+2i}$ such that $X_- m_{\lambda_0+2i} = 0$, when $i = 0, \dots, -\lambda_0$. In fact, $m_{\lambda_0+2i} = (1 \otimes X_+ - (\lambda_0 + 2(i-1))x)m_{\lambda_0+2(i-1)}$. This results in a filtration by \mathfrak{g}_R -modules

$$\mathcal{D}(\mathfrak{g}_R)m_{-\lambda_0} \subset \dots \subset \mathcal{D}(\mathfrak{g}_R)m_{\lambda_0+2i} \subset \mathcal{D}(\mathfrak{g}_R)m_{\lambda_0+2(i-1)} \subset \dots \subset \mathcal{D}(\mathfrak{g}_R)m_{\lambda_0} = M.$$

The successive quotients are

$$\frac{\mathcal{D}(\mathfrak{g}_R)m_{\lambda_0+2(i-1)}}{\mathcal{D}(\mathfrak{g}_R)m_{\lambda_0+2i}} \cong R\mu_i, \quad i = 0, \dots, -\lambda_0,$$

as \mathfrak{g}_R -modules, where $(1 \otimes X_-)\mu_i = 0$, $(1 \otimes H)\mu_i = (\lambda_0 + 2(i-1))\mu_i$ and $(1 \otimes X_+)\mu_i = \frac{1}{2}(\lambda_0 + 2(i-1))x\mu_i$. Since R is simple, it follows that $R\mu_i$ is also simple as \mathfrak{g}_R -module.

Question 2.17. Let \mathfrak{g} be any finite-dimensional Lie algebra over an algebraically closed field k and V be a finite-dimensional \mathfrak{g} -module. Let $\mathfrak{g} \rightarrow T_R$ be a homomorphism of k -Lie algebras such the induced map $\mathfrak{g} \rightarrow k \otimes_R T_R$ is surjective. Let $\mathfrak{g}_R = R \otimes_k \mathfrak{g}$ be the associated transitive Lie algebroid over R . Is then the \mathfrak{g}_R -module $R \otimes_k V$ a local system? Note that R itself is a local system since it is simple over \mathfrak{g}_R , so we may ask more generally: if L is a local system over \mathfrak{g}_R and V a simple \mathfrak{g} -module of finite-dimension, what is $\ell_{\mathfrak{g}_R}(L \otimes_k V)$?

Proposition 2.18. Assume that R is either a formal power series ring over k or a ring of convergent power series over $k = \mathbf{C}$ or $k = \mathbf{R}$. Let \mathfrak{g}_R be a transitive Lie algebroid, i.e. $\alpha(\mathfrak{g}_R) = T_R$, and M be a \mathfrak{g}_R -module of finite type over R . Then

$$(1) \quad \mathfrak{g}_R \cong \mathfrak{g} \rtimes T_R.$$

$$(2) \quad M = RM^{T_R} = R \otimes_k M/\mathfrak{m}_R M, \text{ where in a choice of split in (1) the subalgebroid } T_R \text{ acts trivially on } M/\mathfrak{m}_R M.$$

Proof. (1): In the sequence (2.1), where $\bar{\mathfrak{g}}_R = T_R$, all R -modules are free, so there exists an R -linear split $\phi_0 : T_R \rightarrow \mathfrak{g}_R$. It gives rise to a map $\omega : \wedge^2 T_R \rightarrow \mathfrak{g}$ defining a 2-cocycle in the de Rham complex $\Omega_R^\bullet(\mathfrak{g})$ (the curvature). Since

Poincaré's lemma holds when R is either complete or a ring of convergent power series, there exists an R -linear map $\eta : T_R \rightarrow \mathfrak{g}$ such that $\omega = d\eta$. The map $\phi = \phi_0 - \eta$ then defines an integrable connection on \mathfrak{g}_R .

(2): This is standard, but we still include a proof. Let (x_1, \dots, x_d) be a regular system of parameters of \mathfrak{m}_R , and let ∂_i be an R -basis of T_R such that $\partial_i(x_j) = \delta_{ij}$. We contend that there exist free generators m_i of M such that $\partial_i \cdot m_j = 0$. We know that M is free over R , so let $m' = (m'_1, \dots, m'_n)$ be a column matrix of free generators, so $\partial_d m = \Phi m$ for some $n \times n$ -matrix with coefficients in R . Write

$$\Phi = \sum_{i \geq 0} x_d^i \Phi_i$$

where the Φ_i are independent of x_d , and put

$$\psi = \exp\left(-\sum_{i \geq 0} \frac{x_d^{i+1}}{i+1} \Phi_i\right).$$

This series converges to a nonsingular matrix when R is the ring of convergent power series $k\{x_1, \dots, x_d\}$ ($k = \mathbf{R}$ or $k = \mathbf{C}$) or the ring of formal power series $k[[x_1, \dots, x_d]]$, since the exponent is divisible by x_d . Putting $m = \Psi m'$ we get our basis of M , satisfying $\partial_d m = 0$, i.e. we have free generators of M that belong to M^{∂_d} . The space M^{∂_d} is a module over the Lie algebroid T_{R^0} where $R^0 = R^{\partial_d} = k\{x_1, \dots, x_{d-1}\}$ (resp. $k[[x_1, \dots, x_{d-1}]]$). Putting $N = \sum R^0 m_i \subset M^{\partial_d}$ we have a module T_{R^0} of finite type over the T_{R^0} -simple module R . An induction over $d = \dim R$ completes the proof that M has free T_R -invariant generators. \square

Remark 2.19. Let M be a \mathfrak{g}_A -module of finite length and of finite type over the allowed local ring A . The completed \hat{A} -module $\hat{M} = \hat{A} \otimes_A M$ is then a module over the completed Lie algebroid $\hat{\mathfrak{g}}_{\hat{A}} = \hat{A} \otimes_A \mathfrak{g}_A$. In general, the length increases upon completion: $\ell_{\mathfrak{g}_A}(M) \leq \ell_{\hat{\mathfrak{g}}_{\hat{A}}}(\hat{M})$.

2.4. Ideals of definition

Definition 2.20. Let $(A/k, \mathfrak{g}_A)$ be a Lie algebroid and M a \mathfrak{g}_A -module. An ideal of definition relative to M is an ideal $I \subset A$, $I \neq A$, such that $\alpha(\mathfrak{g}_A) \subset T_A(I)$ and $\ell_{\mathfrak{g}_A}(M/IM) < \infty$.

If $\alpha(\mathfrak{g}_A) = 0$, then an ideal I of definition (relative to A) is the same as the “classical” one, that $\mathfrak{m}_A^l \subset I \subset \mathfrak{m}_A$ for some integer $l \geq 1$ for some maximal ideal \mathfrak{m}_A . If we simply say I is a defining ideal we mean I is an ideal of definition relative to the \mathfrak{g}_A -module A . That defining ideals relative to A exist follows from Zorn's lemma, and in fact there exist defining ideals for any \mathfrak{g}_A -module of finite type over A .

Proposition 2.21. *Let M be a \mathfrak{g}_A -module of finite type over A .*

- (1) *If I is an ideal of definition relative to A , then I is an ideal of definition relative to M .*
- (2) *If I is an ideal of definition relative to M , then $\ell_{\mathfrak{g}_A}(M/I^{n+1}M) < \infty$ for any positive integer n .*

Proof. (1): Put $B = A/I$ and $\mathfrak{g}_B = B \otimes_A \mathfrak{g}_A$, so \mathfrak{g}_B is a Lie algebroid over B and B is simple over \mathfrak{g}_B . Then apply Proposition 2.10, (3), to the \mathfrak{g}_B -module M/IM .

(2): Put $J = \text{Ann } M/IM$, $A_1 = A/J$, then by Proposition 2.10, (4), $\ell_{\mathfrak{g}_A}(A_1) < \infty$. Since $I^i M/I^{i+1}M$ is a module over $\mathfrak{g}_{A_1} = \mathfrak{g}_A/J\mathfrak{g}_A$, it follows by Proposition 2.10, (3), that $\ell_{\mathfrak{g}_A}(I^i M/I^{i+1}M) = \ell_{\mathfrak{g}_{A_1}}(I^i M/I^{i+1}M) < \infty$, $i = 0, 1, 2, \dots$, implying $\ell_{\mathfrak{g}_A}(M/I^{n+1}M) < \infty$, $n = 0, 1, 2, \dots$. \square

If $I \subset J$ are \mathfrak{g}_A -preserved ideals and I is defining, then J is also defining. In particular, if I^n is defining for some positive integer n , then I is defining, and by Proposition 2.21, if I is a defining ideal, then I^n is a defining ideal for any positive integer n . Combining these observations we see that if J is defining and $J^n \subset I$ for some positive integer, then I is also defining, so in particular if $\sqrt{I} = \sqrt{J}$, then I is defining if and only if J is defining. Note however that if $\text{Char } A > 0$, then \sqrt{I} need not be preserved by \mathfrak{g}_A even if I is preserved. If A is a local ring, there exists a unique maximal defining ideal.

When a \mathfrak{g}_A -module of finite type is not finitely generated over A there need not exist a defining ideal. For example, the Weyl algebra $M = k \langle x, \partial_x \rangle$ is a module over the Lie algebroid $T_{k[x]/k}$ (by left multiplication) that lacks a defining ideal. On the other hand, if $\ell_{\mathfrak{g}_A}(M) < \infty$ then any preserved ideal is defining, and if $\ell_{\mathfrak{g}_A}(A) < \infty$ and M is of finite type over A , then $\ell_{\mathfrak{g}_A}(M) < \infty$ (see Proposition 2.10, (3-4)).

Proposition 2.22. *Let A/k be an allowed local ring and \mathfrak{g}_A a Lie algebroid over A .*

- (1) *If J is a maximal \mathfrak{g}_A -defining ideal relative to A , then A/J is a regular ring, hence if A is also a regular ring, then $J = (x_1, \dots, x_r)$, where x_1, \dots, x_r are elements in \mathfrak{m}_A , is a subset of a regular system of parameters of A .*
- (2) *If I is a \mathfrak{g}_A -defining ideal in a local ring A , then its radical $J = \sqrt{I}$ is the unique maximal defining ideal containing I .*

Proof. (1): By Proposition 2.7, A/J is a regular ring. If A is also regular it follows that J is generated by subset of a regular system of parameters.

(2): This follows since $A/I/\text{nil}(A/I)$ is a regular ring (Prop. 2.7). \square

2.4.1. Stratifications

Let X/k be a k -scheme such that the stalk $\mathcal{O}_{X,x}$ of the structure sheaf \mathcal{O}_X is an allowed ring at all points x in X ; denote by $\mathfrak{m}_{X,x}$ the maximal ideal of $\mathcal{O}_{X,x}$, $k_{X,x} = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ its residue field, and let $\text{ht}(x) = \dim \mathcal{O}_{X,x}$ be the height of x . We regard \mathcal{O}_X as a module over the Lie algebroid T_X formed by the sheaf of k -linear tangent vector fields on X , and we say that a point x is preserved (by T_X) if $T_{X,x} \cdot \mathfrak{m}_{X,x} \subset \mathfrak{m}_{X,x}$, so we have a homomorphism $f_x : T_{X,x} \rightarrow T_{k_{X,x}/k}$. Let $\mathcal{P} \subset X$ be the subset of preserved points in X . A *defining point* x is a preserved point such that f_x is surjective. We let $\mathcal{I} \subset \mathcal{P}$ be the subset of defining points in X . For $\xi \in \mathcal{I}$, we put

$$D(\xi) = \{x \in X \mid x \in \xi^-, x \notin \eta^- \text{ when } \text{ht}(\eta) \geq \text{ht}(\xi), \eta \in \mathcal{I}, \eta \neq \xi\}$$

(ξ^- denotes the closure of the point ξ). This results in a stratification of the set of closed points $X^{cl} \subset X$ into a disjoint union of the closed points in regular locally closed subschemes satisfying the frontier condition:

- (i) $X^{cl} = \bigcup_{\xi \in \mathcal{I}} D(\xi)^{cl}$, $D(\xi)$ is a smooth subscheme of X , and if $D(\xi) \cap D(\eta) \neq \emptyset$ for some $\xi, \eta \in \mathcal{I}$, then $D(\xi) = D(\eta)$.
- (ii) If $D(\xi) \cap \bar{D}(\eta) \neq \emptyset$, for some $\xi, \eta \in \mathcal{I}$, then $D(\xi) \subset \partial D(\eta)$.

When $X = V(f)$, $f \in A$, and A is a regular ring this gives an algebraic version of the logarithmic stratification of divisors in [25]. In fact, if \mathfrak{g}_X is any Lie subalgebroid of T_X , which we may think of as being generated by an “involutive system of vector fields”, we also have a similar notion of defining points, resulting in an algebraic version of integral manifolds, which in the analytic case is given by Frobenius’ theorem. A detailed account of the above stratification will be presented elsewhere.

We remark also that the locus of defining points in X can sometimes be regarded as a deformation space for a family of some geometric object.

Example 2.23. The Lie subalgebroid $\mathfrak{g}_A = A\partial_x + A\partial_y \subset T_A$, where $A = \mathbb{C}[x, y, z]$ has the defining points $(z - \lambda) \in \text{Spec } A$, $\lambda \in k$. These planes also form the foliation of \mathbb{C}_k^3 given by the integral manifolds of the involutive system of vector fields $\{\partial_x, \partial_y\}$. On the other hand, if $\delta \in T_A$ does not have any invariant ideals in A , then the stratification of \mathbb{C}^3 consists of just one stratum. On the other hand, if \mathbb{C}^3 is regarded as a complex analytic manifold, then there are plenty of integral manifolds for δ .

Example 2.24. Put $I = (xy(x+y)(x+yz))$, and $X = V(I) \subset \mathbb{A}^3$. The tangential Lie algebroid $T_{\mathbb{A}_k^3}(I)$ is free, which is easily seen using Saito’s criterion (we thank M. Granger for pointing this out), so that the ‘obvious’ tangential derivations form a basis

$$T_{\mathbb{A}_k^3}(I) = k[x, y, z](x\partial_x + y\partial_y) \oplus k[x, y, z](x+y)(y\partial_y - z\partial_z) \oplus k[x, y, z](x+yz)\partial_z,$$

and the tangent sheaf T_X is the sheaf that is associated to the module $T_{\mathbb{A}_k^3}(I)/IT_{\mathbb{A}_k^3}$. The singular locus is $X_s = V(x, y) \cup V(x, y+z) \cup V(y, x+z) \cup V(x+y, x+yz) \subset X$. Let $\xi_1 = (x), \xi_2 = (y), \xi_3 = (x+yz)$ be the generic points of X , and $\eta_1 = (x, y), \eta_2 = (x, y+z), \eta_3 = (y, x+z), \eta_4 = (x+y, x+yz)$ the generic points of X_s , and for any $\lambda \in k$ we put $\eta_\lambda = (x, y, z-\lambda)$. Then the set of preserved points is $\mathcal{P} = \{\xi_i, \eta_i, (\eta_\lambda)_{\lambda \in k}\}$ and the defining points are $\mathcal{I} = \{\xi_i, \eta_2, \eta_3, \eta_4, (\xi_\lambda)_{\lambda \in k}\}$, so η_1 is not a defining point. The stratification of X^{cl} is the disjoint union of the following locally closed smooth subvarieties. Firstly, $D(\xi_i)^{cl}$ is the union of all smooth and closed points in the corresponding component of X , secondly $D(\eta_i)$, $i = 2, 3, 4$, is the union of the smooth points of the components of X_s , except the z -axis, and finally $D(\eta_\lambda)$ run over the closed points on the z -axis; the latter correspond to the “non-holonomic” points in X , in the terminology of [25].

The projection $X \rightarrow \text{Spec } k[z]$ defines a flat family of curve singularities such that no tangent vector field on $\text{Spec } k[z]$ can be lifted to a tangent vector field on X .

2.5. Noetherian invariant rings and finitely generated invariant modules

2.5.1. Semi-simple modules

Let (A, \mathfrak{g}_A) be a Lie algebroid over a k -algebra A . By a *graded \mathfrak{g}_A -algebra* we mean a graded commutative A -algebra $S^\bullet = \bigoplus_{i \geq 0} S^i$, which at the same time is

a \mathfrak{g}_A -module by a homomorphism of A -modules and Lie algebras $\phi : \mathfrak{g}_A \rightarrow T_{S^\bullet}$, such that $\phi(\delta)(S^i) \subset S^i$, $\delta \in \mathfrak{g}_A$.

A graded $(S^\bullet, \mathfrak{g}_A)$ -module is a graded S^\bullet -module and \mathfrak{g}_A -module $M^\bullet = \bigoplus_{i \in \mathbb{Z}} M^i$ such that $\delta \cdot M^i \subset M^i$ and $\delta(sm) = \delta(s)m + s\delta m$, $s \in S^\bullet$, $m \in M^\bullet$, $\delta \in \mathfrak{g}_A$. We let $\text{Mod}(S^\bullet, \mathfrak{g}_A)$ be the category of graded $(S^\bullet, \mathfrak{g}_A)$ -modules M^\bullet which are of finite type over S^\bullet .

When $M \in \text{Mod}(S^\bullet, \mathfrak{g}_A)$ we denote its invariant space $\bar{M}^\bullet = (M^\bullet)^{\mathfrak{g}_A} = \{m \in M \mid \delta \cdot m = 0, \delta \in \mathfrak{g}_A\}$. Clearly, $\bar{S}^\bullet = \bigoplus \bar{S}^i$ is a graded subring of S^\bullet , and \bar{M}^\bullet is an \bar{S}^\bullet -module.

The following theorem generalizes Hilbert's theorem about the finite generation of invariant rings with respect to semi-simple Lie algebras [12].

Theorem 2.25. *Assume that $M^\bullet \in \text{Mod}(S^\bullet, \mathfrak{g}_A)$ where S^\bullet is a graded noetherian \mathfrak{g}_A -algebra.*

- (1) *Assume that S^\bullet is semi-simple over \mathfrak{g}_A and \bar{S}^0 is noetherian. Then \bar{S}^\bullet is a graded and noetherian subring of S^\bullet .*
- (2) *Assume (1) and also that M^\bullet is semi-simple over \mathfrak{g}_A . Then \bar{M}^\bullet is of finite type over \bar{S}^\bullet .*

Lemma 2.26. *Make the assumptions in Theorem 2.25, (1) and (2), so in particular $S^\bullet = A\bar{S}^\bullet \oplus Q$, where Q is a semi-simple \mathfrak{g}_A -submodule of S^\bullet , and $M^\bullet = A\bar{M}^\bullet \oplus M_1$. Then $Q\bar{M}^\bullet \subset M_1$.*

Proof. Put $\mathcal{D} = \mathcal{D}(\mathfrak{g}_A)$ and let $q \in Q$, $\bar{m} \in \bar{M}^\bullet$. We have $q\bar{m} = m_0 + m_1$, where $m_0 \in A\bar{M}^\bullet$ and $m_1 \in M_1$, and our goal is to prove $m_0 = 0$. Since Q is semi-simple it suffices to prove this when $\mathcal{D} \cdot q$ is simple. Since $\mathcal{D} \cdot m_0 \subset A\bar{M}^\bullet$ it follows that $\mathcal{D}m_0 = \mathcal{D} \cdot (\mathcal{D}m_0)^{\mathfrak{g}_A}$. Therefore, since $(\mathcal{D} \cdot q) \cap A\bar{S}^\bullet = \{0\}$, it follows that $\text{Hom}_{\mathcal{D}}(\mathcal{D}m_0, \mathcal{D}q) = 0$; hence $\text{Ann}_{\mathcal{D}}(m_0) \not\subset \text{Ann}_{\mathcal{D}}(q)$, so there exists an element P in \mathcal{D} such that $Pq \neq 0$ and $Pm_0 = 0$. We have now

$$\mathcal{D} \cdot (m_0 + m_1) = \mathcal{D}(q\bar{m}) = (\mathcal{D} \cdot q)\bar{m} = (\mathcal{D} \cdot P \cdot q)\bar{m} = \mathcal{D}P(m_0 + m_1) = \mathcal{D}Pm_1.$$

Therefore $m_0 \in M_1 \cap A\bar{M}^\bullet = \{0\}$. \square

Proof. We first make the following observation:

- (*) If N is an (S^0, \mathfrak{g}_A) -module of finite type over S^0 and semi-simple over \mathfrak{g}_A , then \bar{N} is of finite type over \bar{S}^0 .

Proof: The S^0 -submodule $S^0\bar{N} \subset N$ is of finite type over S^0 , hence there exists an integer n and a surjective homomorphism of (S^0, \mathfrak{g}_A) -modules $\bigoplus_{i=1}^n S^0 \rightarrow S^0\bar{N}$. Since S^0 and N are semi-simple, and therefore also $S^0\bar{N}$ is semi-simple, it follows that this homomorphism is split, so applying the functor $\text{Hom}_{\mathfrak{g}_A}(A, \cdot)$ the induced map $\bigoplus_{i=1}^n \bar{S}^0 \rightarrow (S^0\bar{N})^{\mathfrak{g}_A} = \bar{N}$ is also surjective.

(1): Since S^\bullet is graded noetherian there exists an integer r such that the graded \bar{S}^0 -submodule of $V = \bigoplus_{i=1}^r \bar{S}^i \subset \bar{S}_+^\bullet$ generates $S^\bullet \bar{S}_+^\bullet$ over S^\bullet . Let B^\bullet be the subalgebra of \bar{S}^\bullet that is generated by V and \bar{S}^0 , so in particular $B^i = \bar{S}^i$ when $0 \leq i \leq r$. Let $d > r$ be an integer and assume by induction that $\bar{S}^i = B^i$ when $i < d$. We have (as detailed below)

$$\bar{S}^d = \left(\sum_{1 \leq i \leq r} S^i \cdot \bar{S}^{d-i} \right) \cap \bar{S}^d = \sum_{1 \leq i \leq r} \bar{S}^i \cdot \bar{S}^{d-i} = \sum_{1 \leq i \leq r} B^i \cdot B^{d-i} = B^d.$$

The first equality follows from the inclusion $\bar{S}^d \subset S^\bullet \bar{S}_+ = S^\bullet \cdot V$, noting that V is concentrated in degrees 1 to r . To see clearly the second equality, we first have by Lemma 2.26 that $\bar{S}^{d-i} Q^i \subset Q^d$; then since $S^d = A\bar{S}^d \oplus Q^d$, if $f \in Q^d$, $g \in A\bar{S}^d$, and $f + g \in \bar{S}^d$, it follows that $f = 0$. The last equality follows by induction. This proves that $B^\bullet = \bar{S}^\bullet$. Since $B^0 = \bar{S}^0$ is noetherian and V is of finite type over B^0 by (*), Hilbert's basis theorem implies that \bar{S}^\bullet is finitely generated.

(2): The proof is similar to (1). Since M^\bullet is noetherian there exists an integer r such that the graded \bar{S}^0 -submodule $W = \bigoplus_{1 \leq i \leq r} M^i$ of \bar{M}^\bullet generates the graded S^\bullet -module $S^\bullet \bar{M}_{>0}^\bullet \subset M^\bullet$. There exists also an integer t such that $M^i = 0$ when $i < t$. Let N^\bullet be the \bar{S}^\bullet -submodule of \bar{M}^\bullet that is generated by $\bar{M}_{\leq 0}^\bullet$ and W . Then $N^i = \bar{M}^i$ when $i \leq r$. Let $d > r$ be an integer and assume that $N^i = \bar{M}^i$ when $i < d$. We have then again by induction

$$\bar{M}^d = \left(\sum_{1 \leq i \leq r} S^i \cdot \bar{M}^{d-i} \right) \cap \bar{M}^d = \sum_{1 \leq i \leq r} \bar{S}^i \cdot \bar{M}^{d-i} = \sum_{1 \leq i \leq r} \bar{S}^i \cdot N^{d-i} = N^d.$$

The second equality follows from the inclusion $Q^{d-i} \cdot \bar{M}^i \subset M_1$ (Lem. 2.26). Therefore $\bar{M}^\bullet = N^\bullet$, and since N^\bullet is finitely generated over \bar{S}^\bullet by (*), this completes the proof. \square

Remark 2.27. It is likely that the assumption in Theorem 2.25, (2), can be relaxed so that (1) need not be assumed, at least when A is a field. The problem is that we need to know that the \mathfrak{g}_A -module $\text{Hom}_A(M^\bullet, M^\bullet)$ is semi-simple, since then S^\bullet can be replaced by its image in $\text{Hom}_A(M^\bullet, M^\bullet)$. However, it seems that in general it is a non-trivial problem to see when the \mathfrak{g}_A -module of A -linear homomorphisms of semi-simple modules is semi-simple.

2.5.2. Solvable Lie algebras

Let $(S^\bullet, \mathfrak{g}_k)$ be a noetherian graded \mathfrak{g}_k -algebra over k . By Hilbert's theorem, if S^\bullet is semi-simple over \mathfrak{g}_k , then the invariant ring $(S^\bullet)^{\mathfrak{g}_k}$ is again noetherian (Th. 2.25), while if \mathfrak{g}_k is solvable, Nagata constructed a well-known example of invariant ring $(S^\bullet)^{\mathfrak{g}_k}$ which is not noetherian. There is another way of constructing noetherian \mathfrak{g}_k -subalgebras of a graded noetherian \mathfrak{g}_k -algebra when \mathfrak{g}_k is solvable. Let $\text{Ch}(\mathfrak{g}_k) = (\mathfrak{g}_k / [\mathfrak{g}_k, \mathfrak{g}_k])^*$ be the character group. If $\chi \in \text{Ch}(\mathfrak{g}_k)$, we put $S_\chi^\bullet = \{s \in S^\bullet \mid (X - \chi(X))^n s = 0, n \gg 1, X \in \mathfrak{g}_k\}$. Similarly, we define M_χ^\bullet when M^\bullet is any graded $(\mathfrak{g}_k, S^\bullet)$ -module, and put $\text{supp}_{\mathfrak{g}_k} M^\bullet = \{\chi \in \text{Ch}(\mathfrak{g}_k) \mid M_\chi^\bullet \neq 0\}$.

Theorem 2.28. Let $S^\bullet = \bigoplus_{n \geq 0} S^n$ be a noetherian graded \mathfrak{g}_k -algebra, where \mathfrak{g}_k is a solvable Lie algebra and put $C = \text{supp}_{\mathfrak{g}_k} S^\bullet$. Then

$$S^\bullet = \bigoplus_{\chi \in C} S_\chi^\bullet,$$

and C is a commutative sub-semigroup of $\text{Ch}(\mathfrak{g}_k)$ where the binary operation is induced by the ring structure of S^\bullet .

(1) Let Γ be a subsemigroup of C and put $\Gamma^c = C \setminus \Gamma$. Then

$$S_\Gamma^\bullet = \bigoplus_{\chi \in \Gamma} S_\chi^\bullet$$

is a graded \mathfrak{g}_k -algebra. If moreover $\Gamma + \Gamma^c \subset \Gamma^c$, then S_Γ^\bullet is noetherian.

(2) Let M^\bullet be a finitely-generated graded $(\mathfrak{g}_k, S^\bullet)$ -module and put $C_M = \text{supp}_{\mathfrak{g}_k} M^\bullet$. Then

$$M^\bullet = \bigoplus_{\phi \in C_M} M_\phi^\bullet,$$

and C acts on C_M in a natural way. Let Γ be a subsemigroup of C , Φ be a subset of C_M , and put $\Phi^c = C_M \setminus \Phi$. Consider the conditions:

$$(a) \Gamma \cdot \Phi \subset \Phi,$$

$$(b) \Gamma^c \cdot \Phi \subset \Phi^c.$$

Then (a) implies that

$$M_\Phi^\bullet = \bigoplus_{\phi \in \Phi} M_\phi^\bullet$$

is a graded $(S_\Gamma^\bullet, \mathfrak{g}_k)$ -module. If also (b) is satisfied, then M_Φ^\bullet is of finite type over S_Γ^\bullet .

Proof. (1): Since \mathfrak{g}_k is solvable it follows that $S_\chi^\bullet S_{\chi'}^\bullet \subset S_{\chi+\chi'}^\bullet$, implying that C is a sub-semigroup of $\text{Ch}(\mathfrak{g}_k)$, and S_Γ^\bullet is a \mathfrak{g}_k -subalgebra. It remains to prove that S_Γ^\bullet is noetherian. Define a noetherian algebra B^\bullet as in the proof of Theorem 2.25, so $B^i = S_\Gamma^i$ when $0 \leq i \leq r$. Let $d > r$ be an integer and assume that $B^i = S_\Gamma^i$ when $0 \leq i \leq d-1$. Since $\Gamma + \Gamma^c \subset \Gamma^c$ it follows that $S_{\Gamma^c}^\bullet S_\Gamma^\bullet \subset S_{\Gamma^c}^\bullet$. Therefore $S_\Gamma^{d-i} S_{\Gamma^c}^i \subset S_{\Gamma^c}^d$, and we get as before

$$S_\Gamma^d = \left(\sum_{1 \leq i \leq d-1} S^i \cdot S_\Gamma^{d-i} \right) \cap S_\Gamma^d = \sum_{1 \leq i \leq d-1} S_\Gamma^i \cdot S_\Gamma^{d-i} = \sum_{1 \leq i \leq d-1} B^i \cdot B^{d-i} = B^d.$$

This proves by induction that $B^\bullet = S_\Gamma^\bullet$.

(2): The action of C on C_M is induced by the S^\bullet -action on M^\bullet . The proof that M_Φ^\bullet is of finite type over S_Γ^\bullet is analogous to the proof of Theorem 2.25, (2). \square

3. Hilbert series

3.1. Representation algebras

Let \mathfrak{g}_k be a finite-dimensional Lie algebra over the algebraically closed field k . Let \mathfrak{r} be the radical, $\mathfrak{s} = [\mathfrak{g}_k, \mathfrak{r}] = [\mathfrak{g}_k, \mathfrak{g}_k] \cap \mathfrak{r} \subset \mathfrak{r}$ the nilpotent radical (the second equality follows from the existence of Levi subalgebras), and put $Q = (\mathfrak{r}/\mathfrak{s})^*$, which is a subgroup of the character group $(\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}])^*$ (one may note that $(\mathfrak{r}/\mathfrak{s})^* \cong (\mathfrak{g}_k/[\mathfrak{g}_k, \mathfrak{g}_k])^*$ non-canonically). The group algebra $k[Q]$ is the set of functions $Q \rightarrow k$ which has the value 0 for almost all points in Q ; its elements are commonly described by the expressions $\sum a_q q$, designating the function that maps q to a_q . The algebra $k[Q]$ is also an \mathfrak{r} -module by defining $r \cdot \sum a_q q = \sum a_q q(r)q$, and since every element $q \in Q$ satisfies $q(\mathfrak{s}) = 0$, so q extends to a homomorphism of Lie algebras $\mathfrak{g}_k \rightarrow k$, it follows that $k[Q]$ is also a \mathfrak{g}_k -module. Thus $k[Q]$ is a \mathfrak{g}_k -algebra which is semi-simple over \mathfrak{g}_k and contains all 1-dimensional \mathfrak{g}_k -modules with multiplicity 1.

Let $\mathcal{L}_k = \mathfrak{g}_k/\mathfrak{r}$ be the semi-simple quotient and $\mathfrak{h} \subset \mathfrak{b}_k$ a Cartan algebra and Borel subalgebra of \mathcal{L}_k . Let $P_{++} \subset \mathfrak{h}^*$ be the set of integral dominant weights, which is a commutative semisubgroup of \mathfrak{h}^* , and $\{\omega_1, \dots, \omega_l\}$ be the

set of fundamental weights. For a weight $\phi \in P_{++}$ we let L_ϕ be the simple finite-dimensional \mathcal{L}_k -module of highest weight ϕ . Put $L = \bigoplus_{i=1}^l L_{\omega_i}$ and let $S^\bullet(L)$ be the symmetric algebra, which is a graded \mathcal{L}_k -algebra. The length of a weight $\omega = \sum_{i=1}^l m_i \omega_i \in P_{++}$ is $l(\omega) = \sum_{i=1}^l m_i$. In each degree n , the weight space $S^n(L)_\omega$ is 1-dimensional when $l(\omega)$ attains its maximal value n . Therefore there exists an \mathcal{L}_k -submodule $J^n \subset S^n(L)$ such that

$$S^n(L)/J^n \cong \bigoplus_{l(\omega)=n} L_\omega.$$

Moreover, the \mathcal{L}_k -submodule $S^n(L)J^m \subset J^{n+m}$, since $S^n(L)J^m$ contains no weight vectors of length $n+m$. Therefore $J^\bullet = \bigoplus_{m \geq 1} J^m \subset S^\bullet(L)$ is an \mathcal{L}_k -invariant ideal, so $R_{\mathcal{L}_k}^\bullet := S^\bullet(L)/J^\bullet$ is an \mathcal{L}_k -algebra, which clearly is graded and noetherian. By construction, $R_{\mathcal{L}_k}^\bullet$ contains every simple finite-dimensional \mathcal{L}_k -module with multiplicity 1. We note also that $R_{\mathcal{L}_k}^\bullet$ is a \mathfrak{g}_k -algebra such that $\tau \cdot R_{\mathcal{L}_k}^\bullet = 0$. The tensor product of the \mathfrak{g}_k -algebras $k[Q]$ and $R_{\mathcal{L}_k}^\bullet$

$$R_{\mathfrak{g}_k} = k[Q] \otimes_k R_{\mathcal{L}_k}^\bullet$$

is then a semi-simple \mathfrak{g}_k -algebra, where each simple \mathfrak{g}_k -module has multiplicity 1. We call $R_{\mathfrak{g}_k}$ the representation algebra of \mathfrak{g}_k .

Remarks 3.1. (1) Let $(S^\bullet, \bar{\mathfrak{b}}_k)$ be a $\bar{\mathfrak{b}}_k$ -algebra with $\bar{\mathfrak{b}}_k$ a solvable Lie algebra.

Let $\Gamma = \Gamma^1/\Gamma^0$ is a subquotient of $C = \text{supp}_{\bar{\mathfrak{b}}_k} S^\bullet$, where $\Gamma^0 \subset \Gamma^1$ are subsemigroups of C , then $S_{\Gamma^0}^\bullet \subset S_{\Gamma^1}^\bullet$ are $\bar{\mathfrak{b}}_k$ -subalgebras of $(S^\bullet, \bar{\mathfrak{b}}_k)$, and $S_\Gamma^\bullet = S_{\Gamma^1}^\bullet/S_{\Gamma^0}^\bullet$ is again a $\bar{\mathfrak{b}}_k$ -algebra. In particular, if $\bar{\mathfrak{b}}_k$ is the radical of a Lie algebra \mathfrak{g}_k , and $\Gamma^0 \subset \Gamma^1 \subset Q \times P_{++}$, then $R_\Gamma^\bullet = R_{\Gamma^1}^\bullet/R_{\Gamma^0}^\bullet$ is a $\bar{\mathfrak{b}}_k$ -algebra with multiplicity 1 for each $\bar{\mathfrak{b}}_k$ -simple module L_ϕ , $\phi \in \Gamma^1 \setminus \Gamma^0$. It generates a \mathfrak{g}_k -subalgebra of $(R^\bullet, \mathfrak{g}_k)$, denoted again $R_{\Gamma^1/\Gamma^0}^\bullet$, with multiplicity 1 for each \mathfrak{g}_k -simple module L_ϕ , $\phi \in \Gamma^1 \setminus \Gamma^0$.

- (2) The representation algebras of subsemigroups of P_{++} have natural geometric interpretations. Let G be a semi-simple algebraic group over k and \mathfrak{g}_k be its Lie algebra. Let for example $\Gamma = \{n\lambda \mid n = 1, 2, \dots\}$, where $\lambda \in P_{++}$ and R_Γ^\bullet be defined as above. Then

$$\text{Proj } R_\Gamma^\bullet = G/P$$

where P is a parabolic subgroup of G [21]. Representation algebras are studied geometrically in [16].

- (3) The construction of the representation algebra of a split semi-simple Lie algebra is due to Cartan, and therefore one speaks of ‘‘Cartan multiplication’’ in R^\bullet . Generators and relations for R^\bullet are described by Kostant [17]. A version for Lie groups was introduced in [9] under the name ‘‘universal algebra’’. Bernstein, Gelfand, and Gelfand [4] and Gelfand and Zelevinsky [5] have made concrete realisations of representation algebras (a.k.a. ‘‘models’’) for different Lie groups.

3.2. Algebras over solvable Lie algebras

Let $(S^\bullet, \mathfrak{g}_k)$ be a noetherian graded \mathfrak{g}_k -algebra as in Section 2.5.2, so \mathfrak{g}_k is a finite-dimensional solvable Lie algebra. Let M^\bullet be a graded $(S^\bullet, \mathfrak{g}_k)$ -module

of finite type over S^\bullet , so $M^n = 0$ when $n < n_0$ for some integer n_0 . Let $\mathbf{Z}[C]$ be the group ring of the monoid C , and $\mathbf{Z}[C][t]$ the polynomial ring over $\mathbf{Z}[C]$. Let $\mathbf{Z}[C_M][t]$ be the set of functions $f : C_M \times \mathbf{N} \rightarrow \mathbf{Z}$ which take the value 0 for almost all points in $C_M \times \mathbf{N}$. We write $f = \sum_{\phi, n} k_{\phi, n} \phi t^n$ (finite sum) for the function that maps $(\phi, n) \in C_M \times \mathbf{N}$ to the integer $k_{\phi, n}$. We regard f as a polynomial, though we do not have a counterpart of polynomial products since C_M is not a monoid. Similarly, let $\mathbf{Z}[C][[t]]$ be the ring of formal power series with coefficients in the ring $\mathbf{Z}[C]$, and $\mathbf{Z}[C_M][[t]]$ be the set of all functions $C_M \times \mathbf{N} \rightarrow \mathbf{Z}$. The action of C on C_M gives $\mathbf{Z}[C_M][t]$ ($\mathbf{Z}[C_M][[t]]$) a structure of $\mathbf{Z}[C][t]$ -module ($\mathbf{Z}[C][[t]]$ -module), so that $m\chi t^{n'} \cdot f = \sum m k_{\phi, n} (\chi \cdot \phi) t^{n+n'}$.

The equivariant Hilbert series of M^\bullet is

$$H_{M^\bullet}^{eq}(t) = \sum_{n \geq n_0, \chi \in C_M} \dim_k(M_\chi^n) \chi t^n \in \mathbf{Z}[C_M][[t]].$$

Lemma 3.2.

$$H_{M^\bullet}^{eq}(t) = \frac{f(t)}{\prod_{\chi \in R} (1 - \chi t^{n_\chi})^{d_\chi}},$$

for some finite subset $R \subset C$ and integers $d_\chi > 0$. The integers n_χ are determined by choosing generators of S^\bullet that belong to $S_\chi^{n_\chi}$.

Proof. Use the ordinary proof of the rationality of Hilbert series of finitely generated graded modules over commutative noetherian graded algebras, by induction over the number of generators, which are required to be homogeneous relative to the grading $C_M \times \mathbf{N}$ (see [20] of the ordinary case, and for torus actions, see [22]). \square

Remark 3.3. Replacing the solvable Lie algebra by a torus, Renner [22] interpreted the integers d_χ in the rational presentation of $H_{S^\bullet}^{eq}(t)$ in terms of the geometry of $\text{Proj } S^\bullet$ and its line bundle $\mathcal{O}_X(1)$, and also related it to the geometry of $R \subset C$.

Let $\Gamma \subset C$ be a subsemigroup and $\Phi \subset C_M$ a subset such that the natural Γ -action on C_M preserves Φ , $\Gamma \cdot \Phi \subset \Phi$, so we have the $(S_\Gamma^\bullet, \mathfrak{g}_k)$ -module M_Φ^\bullet (see Theorem 2.28). We define the Φ -Hilbert series

$$H_{M^\bullet}^\Phi(t) = \sum_{n \geq n_0} \dim_k(M_\Phi^n) t^n \in \mathbf{Z}[[t]].$$

Recall that $\dim_k(M_\Phi^n) = \ell_{\mathfrak{g}_k}(M_\Phi^n)$ since k is algebraically closed of characteristic 0, and \mathfrak{g}_k is solvable.

Lemma 3.4. Assume that $\Gamma \subset C$ and $\Phi \subset C_M$ satisfy the conditions in Theorem 2.28, (1) and (2). Then

$$H_{M^\bullet}^\Phi(t) = \frac{g(t)}{\prod_{i=1}^d (1 - t^{n_i})}.$$

The integers n_i are determined by the degrees of a choice of homogeneous generators of S_Γ^\bullet .

Proof. By Theorem 2.28 S_Γ^\bullet is noetherian and M_Φ^\bullet is finitely generated. Then the result follows from Hilbert's theorem. \square

For any finite subset $\Omega \subset C_M$ we have a summation map $\int : \mathbf{Z}[\Omega] \rightarrow \mathbf{Z}$, $\sum a_\phi \phi \mapsto \sum a_\phi$. It induces a same noted map $\int : \mathbf{Z}[C_M][[t]] \rightarrow \mathbf{Z}[[t]]$.

Remark 3.5. Clearly $H_{M^\bullet}^\Phi(t) = \int(H_{M_\Phi^\bullet}^{eq}(t))$, but we can see no apparent reason why \int should map the rational function $H_{M_\Phi^\bullet}^{eq}(t)$ to a rational function of the same form.

3.3. $(S^\bullet, \mathfrak{g}_k)$ -modules

The notation in Section 3.1 remains in force. Let $(S^\bullet, \mathfrak{g}_k)$ be a graded noetherian \mathfrak{g}_k -algebra, where \mathfrak{g}_k is a finite-dimensional Lie algebra over k , and M^\bullet be an $(S^\bullet, \mathfrak{g}_k)$ -module of finite type over S^\bullet . Let $\bar{\mathfrak{b}}_k$ be a maximal solvable Lie subalgebra of \mathfrak{g}_k and Γ be a subquotient of the semigroup of C , where $C = \text{supp}_{\bar{\mathfrak{b}}_k} S^\bullet$ and $\Phi \subset C_M = \text{supp}_{\bar{\mathfrak{b}}_k} M$ on which Γ acts (see Section 2.5.2).

Put

$$\ell_{\mathfrak{g}_k}^\Phi(M^n) = \ell_{\mathfrak{g}_k}(M_\Phi^n) = \sum_{\phi \in \Phi} [M^n : L_\phi],$$

where $[M^n : L_\phi]$ is the multiplicity of the simple \mathfrak{g}_k -module L_ϕ whose highest weight is ϕ .

Theorem 3.6. *Let S^\bullet be a noetherian graded \mathfrak{g}_k -algebra and M^\bullet be graded module over $(S^\bullet, \mathfrak{g}_k)$, which is of finite type over S^\bullet . Let Γ be a subsemigroup of C and $\Phi \subset \text{supp}_{\bar{\mathfrak{b}}_k} M^\bullet$ a subset satisfying the conditions in Theorem 2.28, (1) and (2).*

Then the Hilbert series

$$H_{M^\bullet}^{eq, \Phi}(t) = \sum_{n \in \mathbf{Z}, \phi \in \Phi} [M^n : L_\phi] \phi t^n \in \mathbf{Z}[\Phi][[t]]$$

and

$$H_{M^\bullet}^\Phi(t) = \int(H_M^{eq, \Phi}(t)) = \sum_{n \in \mathbf{Z}} \ell_{\mathfrak{g}_k}^\Phi(M^n) t^n \in \mathbf{Z}[[t]]$$

are rational functions of the form

$$(3.1) \quad H_{M^\bullet}^{eq, \Phi}(t) = \frac{f_M^{eq}(t)}{\prod_{\chi \in \Xi} (1 - \chi t^{n_\chi})},$$

$$(3.2) \quad H_{M^\bullet}^\Phi(t) = \frac{f_M(t)}{\prod_{i=1}^r (1 - t^{n_i})},$$

where the polynomials $f_M^{eq}(t) \in \mathbf{Z}[\Phi, t]$ and $f_M(t) \in \mathbf{Z}[t]$, and $\Xi \subset \Gamma$ is a finite subset.

Letting $\Gamma = C$ and $\Phi = C_M$, we note in particular that the full Hilbert series $H_{M^\bullet}^{eq} := H_{M^\bullet}^{eq, C_M}(t)$ and the generating function $H_{M^\bullet}(t) := H_{M^\bullet}^{C_M}(t)$ of the lengths $\ell_{\mathfrak{g}_k}(M^n)$ are rational functions.

Remark 3.7. The assertion concerning $H_{M^\bullet}^{eq, P^{++}}(t)$ was noted in [22] when \mathfrak{g}_k is a semi-simple Lie algebra.

Proof. There exists a Levi subalgebra $\mathcal{L}_k \subset \mathfrak{g}_k$, and a Borel algebra $\mathfrak{b}_k \subset \mathcal{L}_k$ such that $\bar{\mathfrak{b}}_k = \mathfrak{r} + \mathfrak{b}_k$. Put $\mathfrak{n}_k = [\mathfrak{b}_k, \mathfrak{b}_k]$.

Before starting the proof proper we make a remark: The \mathfrak{n}_k -invariant space $(M^n)^{\mathfrak{n}_k}$ of the \mathfrak{g}_k -module M^n is a \mathfrak{b}_k -module, but need not be an \mathfrak{r} -module when the nilpotent radical $\mathfrak{s}_k \neq 0$ and M^n is not semi-simple. Therefore the $((S^\bullet)^{\mathfrak{n}_k}, \mathfrak{b}_k)$ -module $(M^\bullet)^{\mathfrak{n}_k}$ is not provided with a structure of $\bar{\mathfrak{b}}_k$ -module, which is needed for the application of Lemma 3.4. We need to eliminate the disturbance caused by the presence of a non-trivial nilpotent radical.

For that purpose we construct a filtration of the $(S^\bullet, \mathfrak{g}_k)$ -module M^\bullet by $(S^\bullet, \mathfrak{g}_k)$ -submodules, which induces a \mathfrak{g}_k -module composition series in each homogeneous component M^n . Such a filtration can be constructed inductively. There is an integer n_0 such that $M^n = 0$ when $n < n_0$. Let $\cdots \gamma_{n_0}^{i+1} \subset \gamma_{n_0}^i \subset \cdots$ be a composition series of the \mathfrak{g}_k -module M^{n_0} and $\Gamma_{n_0}^{i,\bullet} \subset M^\bullet$ be the $(S^\bullet, \mathfrak{g}_k)$ -submodule that is generated by $\gamma_{n_0}^i$, so $\Gamma_{n_0}^{i,n_0} = \gamma_{n_0}^i$. Beware that we will below reuse the index i several times to simplify the notation. Let $\gamma_{n_0+1}^i$ be a refinement of the filtration $\Gamma_{n_0}^{i,n_0+1}$ into a composition series of M^{n_0+1} , and $\Gamma_{n_0+1}^{i,\bullet} \subset M^\bullet$ be the refinement of the filtration $\Gamma_{n_0}^{i,\bullet}$ that is determined by $\gamma_{n_0+1}^i$. Then $\Gamma_{n_0+1}^{i,\bullet} \subset M^\bullet$ is a filtration by $(S^\bullet, \mathfrak{g}_k)$ -submodules which induces a composition series of the \mathfrak{g}_k -modules M^{n_0} and M^{n_0+1} . Inductively, knowing the filtration $\Gamma_{n_0+r}^{i,\bullet}$ we refine $\Gamma_{n_0+r}^{i,n_0+r+1}$ into a filtration $\gamma_{n_0+r+1}^i$ of M^{n_0+r+1} , and let $\Gamma_{n_0+r+1}^{i,\bullet}$ be the $(S^\bullet, \mathfrak{g}_k)$ -filtration which is the refinement of $\Gamma_{n_0+r}^{i,\bullet}$ that is determined by $\gamma_{n_0+r+1}^i$. The associated graded $(S^\bullet, \mathfrak{g}_k)$ -module

$$\bigoplus_{n \geq n_0} \bigoplus_i \frac{\Gamma_{n_0+r}^{i,n}}{\Gamma_{n_0+r}^{i+1,n}}$$

is semi-simple over \mathfrak{g}_k in degrees $n \leq n_0 + r$. Moreover, the sequence of filtrations $r \mapsto \Gamma_{n_0+r}^{i,\bullet}$ of M^\bullet stabilises in degrees $n \leq n_0 + r$, so we can put

$$G^n(M^\bullet) = \bigoplus_i \lim_{r \rightarrow \infty} \frac{\Gamma_{n_0+r}^{i,n}}{\Gamma_{n_0+r}^{i+1,n}},$$

by which we intend the common module $\bigoplus_i \frac{\Gamma_{n_0+r}^{i,n}}{\Gamma_{n_0+r}^{i+1,n}}$ when r satisfies $n \leq n_0 + r$, and

$$G^\bullet = \lim_{r \rightarrow \infty} \bigoplus_{n \geq n_0} \bigoplus_i \frac{\Gamma_{n_0+r}^{i,n}}{\Gamma_{n_0+r}^{i+1,n}} = \bigoplus_{n \geq n_0} G^n(M^\bullet).$$

Then G^\bullet is a $(S^\bullet, \mathfrak{g}_k)$ -module which is semi-simple over \mathfrak{g}_k . Moreover, if L_ϕ is the simple module corresponding to the highest weight $\phi \in Q \times P_{++}$, we have by a highest weight argument for short exact sequences, noting also that the multiplicity function $[\cdot : L_\phi]$ is additive in short exact sequences, that

$$[M^n : L_\phi] = [G^n : L_\phi] = \dim_k (G^n)_\phi^{\mathfrak{n}_k},$$

where we also take notice that $\mathfrak{s}_k \cdot (G^n)^{\mathfrak{n}_k} = 0$ since G^n semi-simple over \mathfrak{g}_k , so $(G^n)^{\mathfrak{n}_k}$ is a module over $\bar{\mathfrak{b}}_k$; we can therefore define the weight space $(G^n)_\phi^{\mathfrak{n}_k}$. Let U^\bullet be the image of S^\bullet in $\text{End}_k G^\bullet$. Then if $s \in \mathfrak{s}_k$, $u \in U^\bullet$, and $g \in G^\bullet$, we have $(s \cdot u)g = s \cdot (ug) - u(s \cdot g) = sg' = 0$, for some $g' \in G^\bullet$; hence $\mathfrak{s}_k \cdot U^\bullet = 0$, and

therefore $(U^\bullet)^{\mathfrak{n}_k}$ is a $\bar{\mathfrak{b}}_k$ -algebra. We conclude that the Hilbert series $H_{M^\bullet}^{eq}(t)$ coincides with the Hilbert series of the $((U^\bullet)^{\mathfrak{n}_k}, \bar{\mathfrak{b}}_k)$ -module $(G^\bullet)^{\mathfrak{n}_k}$.

Let $R = R_{\mathcal{L}_k}$ and $U(\mathcal{L}_k)$ be the representation algebra and enveloping algebra of \mathcal{L}_k , respectively. Note that if M is a finite-dimensional \mathcal{L}_k -module, then for any \mathcal{L}_k -linear map $f : U(\mathcal{L}_k) \otimes_{U(\mathfrak{n}_k)} k \rightarrow M$, there exist maps $g : U(\mathcal{L}_k) \otimes_{U(\mathfrak{n}_k)} k \rightarrow R$ and $h : R \rightarrow M$ such that $f = h \circ g$, where g is uniquely determined up to multiplication by a constant; hence $\text{Hom}_{\mathcal{L}_k}(U(\mathcal{L}_k) \otimes_{U(\mathfrak{n}_k)} k, M) = \text{Hom}_{\mathcal{L}_k}(R, M)$. Therefore by adjointness

$$(U^\bullet)^{\mathfrak{n}_k} = \text{Hom}_{\mathfrak{n}_k}(k, U^\bullet) = \text{Hom}_{\mathcal{L}_k}(R, U^\bullet) = ((R)^* \otimes_k U^\bullet)^{\mathcal{L}_k} = (R \otimes_k U^\bullet)^{\mathcal{L}_k}$$

(the idea of expressing \mathfrak{n}_k -invariants as \mathcal{L}_k -invariants was used by Roberts [23], then again by Hadziev [9], to extend Hilbert's finiteness theorem). Similarly,

$$(G^\bullet)^{\mathfrak{n}_k} = (R \otimes_k G^\bullet)^{\mathcal{L}_k}.$$

One can note that the grading of R induces a second grading of the invariant ring and module. Note also that the graded dual \mathfrak{g}_k -module $R^* = \bigoplus_{\omega \in P_{++}} L_\omega^*$ is isomorphic to the \mathfrak{g}_k -module R , where the latter moreover is an algebra. By Hilbert's theorem (see Theorem 2.25) it follows that $(U^\bullet)^{\mathfrak{n}_k}$ is a noetherian graded $\bar{\mathfrak{b}}_k$ -algebra and $(G^\bullet)^{\mathfrak{n}_k}$ is a finitely generated graded $((U^\bullet)^{\mathfrak{n}_k}, \bar{\mathfrak{b}}_k)$ -module. Since Γ and Φ satisfy the conditions in Theorem 2.28 it follows also that $((U^\bullet)^{\mathfrak{n}_k}, \bar{\mathfrak{b}}_k)$ is noetherian and $(G^\bullet)^{\mathfrak{n}_k}$ is a $((U^\bullet)^{\mathfrak{n}_k}, \bar{\mathfrak{b}}_k)$ -module of finite type. Lemma 3.2 now implies that the Hilbert series $H_{M^\bullet}^{eq, \Phi}(t) = H_{G^\bullet}^{eq, \Phi}(t)$ is of the form in equation (3.1). Similarly, Lemma 3.4 implies equation (3.2). \square

Example 3.8. Let R^\bullet be the representation algebra of a free subsemigroup $\Gamma \subset C \subset Q \times P_{++}$, so $\Gamma \cong \mathbf{N}^l$ for some integer l . If $\omega_1, \dots, \omega_l$ are free generators of Γ , then

$$\begin{aligned} H_{R^\bullet}^{eq, \Gamma}(t) &= \sum_{n \geq 0} \left(\sum_{\sum_{i=1}^l n_i = n} \sum_{i=1}^l n_i \omega_i \right) t^n = \prod_{i=1}^l \frac{1}{1 - \omega_i t}, \\ H_{R^\bullet}^\Gamma(t) &= \sum_{n \geq 0} \binom{n+l-1}{l-1} t^n = \frac{1}{(1-t)^l}. \end{aligned}$$

Example 3.9. Let $\mathfrak{g}_k = \mathfrak{sl}_2(k)$, so $P_{++} = \{1, 2, \dots\}$ and the representation algebra $R_{\mathfrak{g}_k}^\bullet$ can be identified with the symmetric algebra $S^\bullet = S^\bullet(V)$, where V is the simple 2-dimensional $\mathfrak{sl}_2(k)$ -module. The Hilbert series of the $(S^\bullet, \mathfrak{g}_k)$ -module $M_d^\bullet = S^\bullet(S^d(V))$ is the ordinary Hilbert series of the graded ring

$$(M_d^\bullet)^{\mathfrak{n}_k} = (M_d^\bullet \otimes_k R^\bullet)^{\mathfrak{g}_k} = S^\bullet(S^d(V) \otimes_k V)^{\mathfrak{g}_k},$$

where \mathfrak{n}_k is a maximal nilpotent subalgebra of \mathfrak{g}_k . Hence $(M_d^\bullet)^{\mathfrak{n}_k}$ coincides with the covariant algebra C_d^\bullet of binary forms of degree d ; see [29, §3.3]. The coefficients of the Hilbert series $H_{C_d^\bullet}(t)$ can be determined using the Cayley-Sylvester decomposition of $S^n(S^d(V))$,

$$\dim_k C_d^n = \sum_{e \geq 0} [S^n(S^d(V)) : S^e(V)] = \sum_{e \geq 0} \left(p(n, d; \frac{nd-e}{2}) - p(n, d; \frac{nd-e}{2} - 1) \right),$$

where $p(d, n; m)$ is the number of partitions of size m inside the rectangle $d \times n$. The rational representation of Hilbert series of invariant algebras is a classical problem; a formula for the rational function $H_{S^\bullet(W)}(t)$ for simple $\mathfrak{sl}_2(k)$ -modules W is presented in [30], and computations for some non-simple ones can be found in [1]. In [29, §3.4] one can find descriptions of the algebra C_d^\bullet for low d . For example:

$C_2^\bullet = k[x_1, x_2]$ where x_1 and x_2 are algebraically independent elements of degree 1 and 2, respectively. We get

$$H_{M_2^\bullet}(t) = \sum_{n \geq 0} ([\frac{n}{2}] + 1)t^n = \sum_{n \geq 0} (\frac{n}{2} - \frac{1}{2} |\cos(\frac{(n+1)\pi}{2})| + 1)t^n = \frac{1}{(1-t)(1-t^2)},$$

where $[\cdot]$ denotes the integer part.

$C_3^\bullet = k[x_1, x_2, x_3, x_4]/(x_1^2 + x_2^3 + x_3^2x_4)$, where $\deg x_1 = 3$, $\deg x_2 = 2$, $\deg x_3 = 1$, $\deg x_4 = 4$. The Hilbert series of the $(S^\bullet(V), \mathfrak{g}_k)$ -module M_3^\bullet can now be computed:

$$H_{M_3^\bullet}(t) = \sum_{n \geq 0} (\frac{n^2}{8} + \frac{n}{2} + \frac{3}{16}(-1)^n + \frac{1}{4} \cos(\frac{n\pi}{2}) + \frac{9}{16})t^n = \frac{t^2 - t + 1}{(1-t)^2(1-t^4)}.$$

We thank L. Bedratyuk for providing us with the Hilbert quasi-polynomial for C_3^\bullet .

3.4. Local systems

Let A be an allowed local k -algebra and $(S^\bullet, \mathfrak{g}_A)$ a finitely generated graded \mathfrak{g}_A -algebra.

Theorem 3.10. *Let $(S^\bullet, \mathfrak{g}_R)$ be a graded \mathfrak{g}_R -algebra where R is simple as \mathfrak{g}_R -module, and let M^\bullet be a $(S^\bullet, \mathfrak{g}_R)$ -module of finite type over S^\bullet . Assume that each $M^n \in \text{Loc}(\mathfrak{g}_R)$. Then the Hilbert series*

$$H_{M^\bullet}(t) = \sum \ell_{\mathfrak{g}_R}(M^n)t^n \in \mathbf{Z}[[t]]$$

is a rational function of the form

$$H_{M^\bullet}(t) = \frac{f_M(t)}{\prod_{i=1}^r (1 - t^{n_i})}$$

where $f_M(t) \in \mathbf{Z}[t]$.

We abstain from writing down the straightforward Γ -equivariant generalisations that can be deduced from Theorem 3.6.

Proof. Apply the functor $k \otimes_R \cdot$ and Theorem 3.6, and keep in mind that the functor $k \otimes_R \cdot$ is exact on the category of \mathfrak{g}_R -modules that are of finite type over R (Prop. 2.6). \square

Note that if J is a maximal defining ideal in an allowed k -algebra A , and N is a \mathfrak{g}_A -module of finite length, then $\ell_{\mathfrak{g}_A}(N) = \ell_{\mathfrak{g}_R}(G_J^\bullet(N))$, where $G_J^\bullet(N) = \oplus_{i \geq 0} J^i N / J^{i+1} N$ and $\mathfrak{g}_R = \mathfrak{g}_A / J \mathfrak{g}_A$.

Theorem 3.11. *Let A be an allowed local k -algebra and \mathfrak{g}_A be a Lie algebroid over A . Let M be a \mathfrak{g}_A -module of finite type as A -module, I be a defining ideal, and put $M^\bullet = \bigoplus_{i \geq 0} I^i M / I^{i+1} M$. Let $J = \sqrt{I}$ be the maximal defining ideal of A and put $R = A/J$ and $\mathfrak{g}_R = \mathfrak{g}_A / J\mathfrak{g}_A$, which is a Lie algebroid over R . Assume that $G_J^\bullet(M^n) \in \text{Loc}(\mathfrak{g}_R)$ for each integer $n, i = 0, 1, 2, \dots$*

(1) *The Hilbert series*

$$H_M(t) = \sum_{n \geq 0} \ell_{\mathfrak{g}_A}(M^n) t^n \in \mathbf{Z}[[t]]$$

is a rational function of the form

$$H_M(t) = \frac{f_M(t)}{\prod_{i=1}^r (1 - t^{n_i})},$$

where $f_M(t) \in \mathbf{Z}[t]$.

(2) *The length function*

$$n \mapsto \chi_M^I(n) = \ell_{\mathfrak{g}_A}(M/I^{n+1}M).$$

is determined by a quasi-polynomial $\phi_M^I(t)$ with integer coefficients for high n . The degree and leading coefficient of this quasi-polynomial are well-defined numbers, so

$$\phi_M^I(t) = \frac{e}{d!} n^d + g(t)$$

where $g(t)$ is a quasi-polynomial of degree at most $d - 1$.

Remark 3.12. (1) We do not know if it suffices that $J^n I^i M / (J^{n+1} I^i M + I^{i+1} M)$ are local systems for finitely many n, i to conclude that they are local systems for all n, i .

(2) The condition that $J^n I^i M / (J^{n+1} I^i M + I^{i+1} M)$ are local systems for all n, i is satisfied when A is an analytic algebra (Prop. 2.18).

Proof. (1) follows from Theorem 3.10. It is well-known that (1) implies (2). For a discussion why the quasi-polynomial has a well-defined degree d and leading coefficient e , see [6]. \square

Definition 3.13. Let (A, \mathfrak{g}_A) , M , and I be as in Theorem 3.11. The dimension $d_{\mathfrak{g}_A}(M, I)$ and multiplicity $e_{\mathfrak{g}_A}(M, I)$ of M is the integer d and rational number e in Theorem 3.11, (2).

The number $d_{\mathfrak{g}_A}(M, I)$ is independent of the choice of defining ideal I . To see this, let I' be any other defining ideal. Then $I_1 = I + I'$ is a defining ideal and $I \subset I_1$. Since $l_{\mathfrak{g}_A}((I_1^{n+1} + I)/I) < l_{\mathfrak{g}_A}((I_1^n + I)/I)$ as long as $(I_1^n + I)/I \neq 0$, it follows that $I_1^n \subset I \subset I_1$ when $n \gg 1$. Therefore there exist positive integers a and b such that $I^a \subset I'$ and $I'^b \subset I$. From this the assertion follows. Thus we can write $d_{\mathfrak{g}_A}(M)$, without reference to a choice of defining ideal.

Remark 3.14. It is straightforward to see that the dimension $d(M)$ and multiplicity $e(I, M)$ have the same properties as expounded in [20, §13].

3.5. Structure of fibre Lie algebras

Recall that for a \mathfrak{g}_k -module M of finite dimension we have

$$\ell_{\mathfrak{g}_k}(M) = \dim_k M^{\mathfrak{n}_k},$$

where \mathfrak{n}_k is a maximal nilpotent subalgebra of a Levi subalgebra of \mathfrak{g}_k , and thus to compute lengths of \mathfrak{g}_k -modules a first step is to find the structure of the Levi factors of \mathfrak{g}_k .

Let $\mathcal{D}^s(\mathfrak{g}_k)$ denote the derived central series of a Lie algebra \mathfrak{g}_k , so $\mathcal{D}^0(\mathfrak{g}_k) = \mathfrak{g}_k$, $\mathcal{D}^s(\mathfrak{g}_k) = [\mathcal{D}^{s-1}(\mathfrak{g}_k), \mathcal{D}^{s-1}(\mathfrak{g}_k)]$, $s = 1, \dots$, and \mathfrak{g}_k is solvable if $\mathcal{D}^s(\mathfrak{g}_k) = 0$ when $s \gg 1$. Since k is algebraically closed of characteristic 0, we have $\ell_{\mathfrak{g}_k}(M) = \dim_k M$ when \mathfrak{g}_k is solvable. The following proposition shows more precisely the relevance of knowing if the fibre Lie algebra of a Lie algebroid is solvable.

Proposition 3.15. *Let \mathfrak{g}_A be a Lie algebroid over a local ring A that preserves the maximal ideal \mathfrak{m}_A . Let J be a defining ideal of the \mathfrak{g}_A -module A and M be a \mathfrak{g}_A -module which is of finite type over A . If the fibre Lie algebra $\mathfrak{g}_k = \mathfrak{g}_A/\mathfrak{m}_A \mathfrak{g}_A$ is solvable, then*

$$\sum_{i \geq 0} \ell_{\mathfrak{g}_A} \left(\frac{J^i M}{J^{i+1} M} \right) t^i = \sum_{i \geq 0} \dim_k \left(\frac{J^i M}{J^{i+1} M} \right) t^i = \frac{f(t)}{(1-t)^{d-1}},$$

where $f \in \mathbf{Z}[t]$.

Proof. If N is an A -module we put $G_{\mathfrak{m}_A}^\bullet(N) = \oplus_{i \geq 0} \mathfrak{m}_A^i N / \mathfrak{m}_A^{i+1} N$. Since \mathfrak{g}_A preserves \mathfrak{m}_A , it follows that \mathfrak{m}_A is the maximal defining ideal. Since $\ell_{\mathfrak{g}_A}(\frac{J^i M}{J^{i+1} M}) = \ell_{\mathfrak{g}_A}(G_{\mathfrak{m}_A}^\bullet(\frac{J^i M}{J^{i+1} M})) = \ell_{\mathfrak{g}_k}(G_{\mathfrak{m}_A}^\bullet(\frac{J^i M}{J^{i+1} M})) = \dim_k(G_{\mathfrak{m}_A}^\bullet(\frac{J^i M}{J^{i+1} M}))$. The last equality in the proposition follows from Hilbert's theorem. \square

Proposition 3.16. *Let \mathfrak{g}_A be a Lie algebroid over a regular allowed local k -algebra A and let J be a maximal defining ideal of A . Put $\mathfrak{b}_A = \text{Ker}(\mathfrak{g}_A \rightarrow T_A)$, and assume that $\mathfrak{b}_k = \mathfrak{b}_A/\mathfrak{m}_A \mathfrak{b}_A$ is a solvable Lie algebra. Then the Levi factors of the fibre Lie algebra \mathfrak{g}_k of $\mathfrak{g} = \text{Ker}(\mathfrak{g}_A/J\mathfrak{g}_A \rightarrow T_A(J)/JT_A)$ act faithfully on $k \otimes_A J/J^2 = J/\mathfrak{m}_A J$. Hence the Levi sub-algebras of \mathfrak{g}_k are determined by the Levi sub-algebras of the image of \mathfrak{g}_k in $\mathfrak{gl}(J/\mathfrak{m}_A J)$.*

Proof. It suffices to see that $\mathfrak{a}_k := \text{Ker}(\mathfrak{g}_k \rightarrow \mathfrak{gl}(J/\mathfrak{m}_A J))$ is solvable. Let $\mathfrak{a} \subset \mathfrak{g} \subset \mathfrak{g}_A/J\mathfrak{g}_A$ be the inverse image of \mathfrak{a}_k for the projection $\mathfrak{g} \rightarrow \mathfrak{g}_k = \mathfrak{g}/\mathfrak{m}_A \mathfrak{g}$, and \mathfrak{a}_A the inverse image of \mathfrak{a} for the projection $\mathfrak{g}_A \rightarrow \mathfrak{g}_A/J\mathfrak{g}_A$. Then \mathfrak{a}_A is a Lie sub-algebroid of \mathfrak{g}_A such that

$$\mathfrak{b}_A \subset \mathfrak{a}_A, \quad \alpha(\mathfrak{a}_A) \subset JT_A, \quad \alpha(\mathfrak{a}_A)(J) \subset \mathfrak{m}_A J.$$

Therefore,

$$\alpha(\mathcal{D}^1(\mathfrak{a}_A)) = [\alpha(\mathfrak{a}_A), \alpha(\mathfrak{a}_A)] \subset \mathfrak{m}_A JT_A \subset \mathfrak{m}_A^2 T_A.$$

Since $[\mathfrak{m}_A^k T_A, \mathfrak{m}_A^k T_A] \subset \mathfrak{m}_A^{2k-1} T_A$, and $\cap_{i \geq 1} \mathfrak{m}_A^i T_A = 0$ by Krull's intersection theorem, we have

$$\cap_{s \geq 1} \mathcal{D}^s(\mathfrak{a}_A) \subset \mathfrak{b}_A.$$

Since \mathfrak{b}_k is solvable there exists a positive integer r such that $\mathcal{D}^r(\mathfrak{b}_k) \subset \mathfrak{m}_A \mathfrak{b}_A$. Therefore

$$\cap_{s \geq 1} \mathcal{D}^s(\mathfrak{a}_A) \subset \mathfrak{m}_A \mathfrak{b}_A \subset \mathfrak{m}_A \mathfrak{a}_A.$$

Since $\dim \mathfrak{a}_k < \infty$ it follows that there exists a positive integer r_1 such that $\mathcal{D}^{r_1}(\mathfrak{a}_A) \subset \mathfrak{m}_A \mathfrak{a}_A$, so $\mathcal{D}^{r_1}(\mathfrak{a}_k) = 0$. \square

Put $V = J/\mathfrak{m}_A J$. By Proposition 3.16, if \mathfrak{b}_k is solvable we can recognize the structure of the Levi factors of \mathfrak{g}_k from its image in $\mathfrak{gl}_k(V)$. A number of results dealing with the problem of identifying Lie sub-algebras of $\mathfrak{gl}_k(V)$ can be found in [15, Ch. 1]. We have use for the following consequence of one of these results.

Theorem 3.17. *Let $\mathfrak{g} \subset \mathfrak{gl}_k(V)$ be a Lie subalgebra with radical $\mathfrak{r} \subset \mathfrak{g}$, and \mathfrak{h} be a Cartan subalgebra of $\mathfrak{sl}_k(V) \subset \mathfrak{gl}_k(V)$. Let $V_{r+1} = 0 \subset V_r \subset V_{r-1} \subset \cdots \subset V_1 \subset V_0 = V$ be a composition series of V as \mathfrak{g} -module, and put $n_i = \dim_k V_i/V_{i+1}$, $i = 0, 1, \dots, r$ be the dimensions of the simple subquotients. Let F be the subset of indices f such that $n_f \geq 2$. Suppose that $\mathfrak{h} \subset \mathfrak{g}$. Then*

$$\mathfrak{g} \cong \mathfrak{r} \rtimes \bigoplus_{f \in F} \mathfrak{sl}_{n_f}.$$

Proof. Put $W_i = V_i/V_{i+1}$, which is a simple \mathfrak{g} -module, hence by Lie's theorem, the canonical homomorphism $\mathfrak{g} \rightarrow \mathfrak{gl}_k(W_i)$ maps \mathfrak{r} to $k \subset \mathfrak{gl}_k(W_i)$. Therefore there exists an injective homomorphism

$$\mathfrak{g}/\mathfrak{r} \rightarrow \bigoplus_{f \in F} \mathfrak{sl}_k(W_f).$$

Let \mathfrak{g}_f be the image of $\mathfrak{g}/\mathfrak{r}$ in $\mathfrak{sl}_k(W_f)$. Since $\mathfrak{h} \subset \mathfrak{g}$, the image of \mathfrak{h} in \mathfrak{g}_f contains a Cartan subalgebra of $\mathfrak{sl}_k(W_f)$. In particular, in the weight basis of W_f , the diagonal matrix $\text{Diag}(n_f - 1, -1, \dots, -1) \in \mathfrak{g}_f$, and as $\mathfrak{g}/\mathfrak{r}$ is semi-simple, hence \mathfrak{g}_f is semi-simple, it follows by Kostant's theorem (see [15, Theorem 1.1]) that $\mathfrak{g}_f = \mathfrak{sl}_k(W_f)$. \square

4. Toral Lie algebroids

We will in this section consider ideals $I = (f_1, \dots, f_s)$ in an allowed regular local k -algebra A , and Lie subalgebroids \mathfrak{g}_A of its tangential derivations $T_A(I)$ (the reader may take $\mathfrak{g}_A = T_A(I)$). There exists a maximal defining ideal J_m of the \mathfrak{g}_A -module A , so that $R = A/J_m$ is a simple module over the Lie algebroid $\mathfrak{g}_R = \mathfrak{g}_A/J_m \mathfrak{g}_A$, and $\ell_{\mathfrak{g}_A}(I/J_m I) < \infty$. We ask ourselves the question when each homogeneous component of the \mathfrak{g}_R -module $G_{J_m}^\bullet(I) = \bigoplus_{n \geq 0} J^n I / J^{n+1} I$ belongs to $\text{Loc}(\mathfrak{g}_R)$. In the analytic situation this is not a problem (Prop. 2.18), but in general it seems difficult to answer. The problem is illustrated by the following proposition, stating that if the generators f_1, \dots, f_s of I can be selected judiciously, then we do get local systems.

Proposition 4.1. *Let \mathfrak{g}_A be a Lie algebroid and $I = (f_1, \dots, f_s)$ an ideal of A such that $\mathfrak{g}_A \cdot I \subset I$. Let $\{x_1, \dots, x_m\}$ be a regular system of parameters such that $J_m = (x_1, \dots, x_r)$. Assume that there exist $\delta_1, \dots, \delta_l \in \mathfrak{g}_A$ that map to generators of the Lie algebroid $\bar{\mathfrak{g}}_R = \text{Im}(\mathfrak{g}_R \rightarrow T_R) \subset T_R$ such that*

$$\delta_j \cdot f_i \equiv \lambda_j f_i \pmod{(x_{r+1}, \dots, x_m) + I \cdot (x_1, \dots, x_r)}, \quad j = 1, \dots, l,$$

where $\lambda_j \in A$ is the same for all i . Then $G_{J_m}^n(I) \in \text{Loc}(\mathfrak{g}_R)$, $n = 0, 1, 2, \dots$. This holds in particular when I is a principal ideal.

Thus if \mathfrak{g}_A and I satisfy the assumption in Proposition 4.1 it follows from Theorem 3.11 that the Hilbert series $H_I(t)$ of the \mathfrak{g}_A -module I is rational. In particular, considering a principal ideal $(f) \subset I$ as a module over its tangent algebra $T_A((f))$, its Hilbert series $H_{(f)}(t)$ is always rational.

Proof. The R -module $G_{J_m}^n(I)$ is generated by the elements $m_{\alpha,i} = X^\alpha f_i \bmod J_m^{n+1}I$, $i = 1, \dots, s$, $|\alpha| = n$, where $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_r^{\alpha_r}$. Put $\bar{\delta}_i = \delta_i \bmod J_m \mathfrak{g}_A \in \mathfrak{g}_R$. Since $\delta_i \in T_A(J_m)$ it follows that $\delta_i \cdot X^\alpha f_i \equiv X^\alpha \delta_i(f_i) \bmod J_m I$, and therefore by assumption, $\bar{\delta}_j \cdot m_{\alpha,i} = \lambda_j m_{\alpha,i} \bmod \mathfrak{m}_R I$. It follows from Proposition 2.15 that $G_{J_m}^n(I) \in \text{Loc}(\mathfrak{g}_R)$. \square

Let \mathbf{q}_A be a Lie sub-algebroid of $T_A(\mathbf{m}_A)$ and \mathbf{q}_k its fibre Lie algebra. Say that \mathbf{q}_A is a *weakly toral* Lie algebroid if the \mathbf{q}_k -module $\mathbf{m}_A/\mathbf{m}_A^2$ satisfies:

- (i) $\ell_{\mathbf{q}_k}(\mathbf{m}_A/\mathbf{m}_A^2) = \dim_k \mathbf{m}_A/\mathbf{m}_A^2$;
- (ii) $\mathbf{m}_A/\mathbf{m}_A^2$ is multiplicity free;
- (iii) $(\mathbf{m}_A/\mathbf{m}_A^2)^{\mathbf{q}_k} = 0$.

It follows that if \mathbf{q}_A is weakly toral, then there exists a regular system of coordinates $\{x_1, \dots, x_n\}$ such that $\mathbf{m}_A/\mathbf{m}_A^2 = \bigoplus_{i=1}^n k\bar{x}_i$, where $x_i \equiv \bar{x}_i \bmod \mathbf{m}_A^2$, and $k\bar{x}_i \not\cong k\bar{x}_j$ as \mathbf{q}_k -modules when $i \neq j$. Say that \mathbf{q}_A is a *toral* Lie algebroid if the derivations $\nabla_{x_i} = x_i \partial_{x_i}$, $i = 1, \dots, n$, are contained in \mathbf{q}_A for some choice of regular system of parameters; clearly, toral Lie algebroids are weakly toral. The advantage of the notion of weakly toral as opposed to toral Lie algebroids, besides being more general, is of course that the former does not refer to a choice of parameters.

Say that an ideal $I \subset A$ is *monomial* if there exists a toral algebra \mathbf{q}_A such that $\mathbf{q}_A \subset T_A(I)$. We have the following characterisation:

Proposition 4.2. *The following are equivalent for an ideal $I \subset A$ and toral Lie algebroids \mathbf{q}_A :*

- (1) *I is a monomial ideal with respect to a toral Lie algebroid \mathbf{q}_A .*
- (2) *There exists a regular system of parameters $\{x_1, \dots, x_n\}$ such that $\{\nabla_{x_1}, \dots, \nabla_{x_n}\} \subset \mathbf{q}_A$ and I is generated by monomials of the form $X^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.*

Proof. (1) \Rightarrow (2): Select a regular system of parameters such that $\{\nabla_{x_1}, \dots, \nabla_{x_n}\} \subset \mathbf{q}_A$. For any integer $r \geq 0$ we can write $f = \sum_{|\alpha| \leq r} f_\alpha X^\alpha + f_r$, where $f_\alpha \in k$ and $f_r \in \mathbf{m}_A^{r+1}$. Let $\text{cont}_r(f)$ be the set of monomials X^α in this expression such that $f_\alpha \neq 0$, put

$$I_r = (\text{cont}_r(f) \mid f \in I),$$

and $\hat{I} = \bigcup_{r \geq 0} I_r$. Since A is noetherian, $\hat{I} = I_r$ for sufficiently high r , hence \hat{I} is generated by certain monomials X^α . We assert that $I = \hat{I}$.

If $f \in I$ and \bar{f} is its projection in $B = A/\hat{I}$, we have by the definition of \hat{I} that $\bar{f} \in \bigcap_{i \geq 1} \mathbf{m}_B^i = \{0\}$, by Krull's intersection theorem, since B is a local noetherian ring. Therefore $f \in \hat{I}$, showing that $I \subset \hat{I}$. To see that $\hat{I} \subset I$, let $f \in I$ and $X^\alpha \in \text{cont}_r(f)$ for some integer r . For any integer $r' \geq r$ there exists a differential operator $P(\nabla)$ in the ∇_{x_i} such that $P(\nabla)(f) = X^\alpha + g$, where $g \in \mathbf{m}_A^{r'+1}$. Since $P(\nabla)(f) \in I$, it follows that $X^\alpha \in \bigcap_{r > 0} (I + \mathbf{m}_A^r) = I$, by Krull's intersection theorem.

(2) \Rightarrow (1): This is evident. \square

Proposition 4.3. *Let \mathbf{q}_A be a weakly toral Lie algebroid in T_A and \mathfrak{g}_A be a Lie algebroid such that $\mathbf{q}_A \subset \alpha(\mathfrak{g}_A)$, and assume that J_m is the maximal \mathfrak{g}_A -defining ideal of A .*

There exists a regular system of parameters $\{x_1, \dots, x_n\}$ such that $J_m = (x_1, \dots, x_r)$ and $\mathfrak{m}_A/\mathfrak{m}_A^2 = \bigoplus_{i=1}^n k\bar{x}_i$ as \mathbf{q}_k -module, the Lie algebroid $\mathfrak{g}_R = \mathfrak{g}_A/(J_m\mathfrak{g}_A)$ over $R = A/J_m$ is transitive, and the following are equivalent:

- (1) $x_i \in J_m$.
- (2) $\partial_{x_i} \notin \alpha(\mathfrak{g}_A)$.

Proof. That J_m is generated by a subset of a regular system of parameters for A , as stated, follows since A and R are regular rings. As \mathfrak{g}_A preserves (x_1, \dots, x_r) , we get that \mathbf{q}_k acts stably on $\sum_{i=1}^r k\bar{x}_i \subset \mathfrak{m}_A/\mathfrak{m}_A^2$. Therefore, by a k -linear change of the parameters $\{x_{r+1}, \dots, x_n\}$ and $\{x_1, \dots, x_r\}$, separately, we get a new regular system of parameters which induces a decomposition of the \mathbf{q}_k -module $\mathfrak{m}_A/\mathfrak{m}_A^2$.

We prove that the image $\bar{\mathfrak{g}}_R$ of the map $\mathfrak{g}_R \rightarrow T_R$ equals T_R by considering the exact sequence $0 \rightarrow \bar{\mathfrak{g}}_R \rightarrow T_R \rightarrow T_R/\bar{\mathfrak{g}}_R \rightarrow 0$. Since R is simple over $\bar{\mathfrak{g}}_R$ and $\bar{\mathfrak{g}}_R$ acts with the adjoint action on the terms in the sequence, it follows that all the terms are free over R (Prop. 2.6), so $\bar{\mathfrak{g}}_R = T_R$ will follow if their ranks agree, i.e. it suffices to see that $\text{rank } \bar{\mathfrak{g}}_R = n - r$. Consider now the \mathbf{q}_k -module $\mathfrak{m}_A/\mathfrak{m}_A^2 = \bigoplus_{i=1}^n k\bar{x}_i$. There exist elements $\bar{\delta}_1, \dots, \bar{\delta}_n \in \mathbf{q}_k$ such that $\bar{\delta}_i \cdot \bar{x}_j = \delta_{ij}$. Let $\delta_i \in \mathfrak{g}_A$ be elements that map to the $\bar{\delta}_i$, and $\hat{\delta}_i$ be their image in $K(R) \otimes_R \bar{\mathfrak{g}}_R$, where $K(R)$ is the fraction field of R . It suffices now to see that $B = \{\hat{\delta}_{r+1}, \dots, \hat{\delta}_n\}$ forms a basis of the vector space $K(R) \otimes_R \bar{\mathfrak{g}}_R$, which will follow if they are linearly independent. Since δ_i maps to $\bar{\delta}_i$ it follows that $\delta_i = u_i x_i \partial_{x_i} + \eta_i$, where $\eta_i \in \mathfrak{m}_A^2 T_A$, and therefore $\hat{\delta}_i = \hat{u}_i x_i \partial_{x_i} + \hat{\eta}_i$, $i = r+1, \dots, n$, where \hat{u}_i is a unit in R and $\hat{\eta}_i \in \mathfrak{m}_R^2 T_R$. Put $S = (\hat{\delta}_i(x_j))_{r+1 \leq i, j \leq n}$. Then

$$\det S = \hat{u}_{r+1} \cdots \hat{u}_n x_{r+1} \cdots x_n + \phi,$$

where $\phi \in \mathfrak{m}_R^{n-r+1}$, so in particular $\det S \neq 0$. This implies that B is a basis of $K(R) \otimes_R \bar{\mathfrak{g}}_R$.

That (1) \Rightarrow (2) is evident since $\mathfrak{g}_A \cdot J_m \subset J_m$. The converse follows since $\mathfrak{g}_R \rightarrow T_R$ is surjective. \square

Theorem 4.4. *Let \mathfrak{g}_A be a Lie algebroid over a regular local allowed algebra A and $I \subset A$ an ideal such that $\mathfrak{g}_A \cdot I \subset I$ (e.g. $I = A$), and put $\mathfrak{b} = \text{Ker}(\mathfrak{g}_A \rightarrow T_A)$. Make the following assumptions:*

- (a) \mathfrak{g}_A contains a Lie sub-algebroid \mathbf{q}'_A such that $\alpha(\mathbf{q}'_A)$ is a weakly toral Lie sub-algebroid of T_A .
- (b) $\mathfrak{b}_k = \mathfrak{b}/\mathfrak{m}_A \mathfrak{b}$ is solvable.

Let J_m be a maximal defining ideal of A , $S^\bullet = \bigoplus_{i \geq 0} J_m^i/J_m^{i+1}$, $R = A/J_m$, $\mathfrak{g}_R = \mathfrak{g}_A/J_m \mathfrak{g}_A$, and $V = J_m/\mathfrak{m}_A J_m$. Let $\mathfrak{g}_k = k \otimes_R \text{Ker}(\mathfrak{g}_R \rightarrow T_R)$ be the fibre Lie algebra of \mathfrak{g}_R .

- (1) *The fibre Lie algebra is the semi-direct product*

$$\mathfrak{g}_k = \mathfrak{r} \rtimes \bigoplus_{f \in F} \mathfrak{sl}_{n_f}$$

where \mathfrak{r} is the radical of \mathfrak{g}_k , and the integers n_f are the dimensions of the simple subquotients of the \mathfrak{g}_k -module V that are of dimension > 1 .

(2) Let \mathfrak{n}_k be a maximal nilpotent subalgebra of a Levi factor of \mathfrak{g}_k . Then

$$(k \otimes_R S^\bullet)^{\mathfrak{n}_k} = S^\bullet(V^{\mathfrak{n}_k}),$$

where $S^\bullet(V^{\mathfrak{n}_k})$ is the symmetric algebra of the \mathfrak{n}_k -invariant subspace of V .

(3) Assume that the Lie algebroid $\alpha(\mathfrak{q}'_A)$ in (a) is toral, and let J be any defining ideal of the \mathfrak{g}_A -module A . The function

$$n \mapsto \chi_I^J(n) = \ell_{\mathfrak{g}_A}(I/J^{n+1}I)$$

is a polynomial for high n . The \mathfrak{g}_A -dimension and \mathfrak{g}_A -multiplicity with respect to J_m (see Definition 3.13) of the \mathfrak{g}_A -module A are

$$d_{\mathfrak{g}_A}(A) = \ell_{\mathfrak{g}_k}(V) \quad \text{and} \quad e_{\mathfrak{g}_A}(A, J_m) = 1.$$

Proof. (1): Put $\mathfrak{g} = \text{Ker}(\mathfrak{g}_R \rightarrow T_R)$, so $\mathfrak{g}_k = k \otimes_R \mathfrak{g}$, and let $\phi : \mathfrak{g}_k \rightarrow \mathfrak{gl}_k(V)$ define the representation of \mathfrak{g}_k in V . By (a) there exists a regular system of parameters as in Proposition 4.3, so $V = \bigoplus_{i=1}^r k\bar{x}_i$. Letting \mathfrak{q}'_R be the image of \mathfrak{q}'_A in \mathfrak{g}_R , the image of $\mathfrak{q}'_R \cap \mathfrak{g}$ under the composed map $\mathfrak{q}'_R \cap \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}_k \rightarrow \mathfrak{gl}_k(V)$ will then contain the commutative Lie algebra $\sum_{i=1}^r k\nabla_{x_j}$. Hence a Cartan subalgebra of $\mathfrak{sl}_k(V)$ is contained in $\phi(\mathfrak{g}_k)$. By Theorem 3.17 it follows that $\phi(\mathfrak{g}_k) = \mathfrak{r}_1 \rtimes \bigoplus_{f \in F} \mathfrak{sl}_{n_f}$, where \mathfrak{r}_1 is the radical of $\phi(\mathfrak{g}_k)$; therefore, since (b) holds, Theorem 3.16 implies (1).

(2): We use the notation in the proof of Theorem 3.17. Put $\mathcal{L}_k = \bigoplus_{f \in F} \mathfrak{sl}_{n_f}$, so $V = \bigoplus_{i=1}^r W_i = V^{\mathcal{L}_k} \oplus \bigoplus_{f \in F} W_f$, where the W_i are simple \mathcal{L}_k -modules. Each symmetric product $S^l(W_f)$ is simple over $\mathfrak{sl}_k(W_f)$, and therefore the \mathcal{L}_k -module $k \otimes_R J_m^n / J_m^{n+1} = S^n(V)$ has the semi-simple decomposition

$$S^n(V) = \bigoplus_{k_i \geq 0, \sum k_i = n} S^{k_1}(W_1) \otimes_k S^{k_2}(W_2) \otimes_k \cdots \otimes_k S^{k_r}(W_r).$$

Therefore, if \mathfrak{n}_k is a maximal nilpotent subalgebra of \mathcal{L}_k , the corresponding highest weight space is

$$S^n(V)^{\mathfrak{n}_k} = \bigoplus_{k_i \geq 0, \sum k_i = n} kX_1^{k_1} \otimes X_2^{k_2} \cdots \otimes X_r^{k_r}$$

where X_i is a basis of $W_i^{\mathfrak{n}_k}$. Since $k[X_1, \dots, X_r] = k[X_1] \otimes_k \cdots \otimes_k k[X_r]$, it follows that $(k \otimes_R S^\bullet)^{\mathfrak{n}_k} = S^\bullet(V^{\mathfrak{n}_k})$.

(3): Since \mathfrak{b} acts trivially on A we can assume that $\mathfrak{g}_A \subset T_A(I)$. Put $N^n = k \otimes_R G_{J_m}^\bullet(J^n I / J^{n+1} I)$ (note that $G_{J_m}^i(J^n I / J^{n+1} I) = J_m^i J^n I / (J_m^{i+1} J^n I + J^{n+1} I)$ is non-zero only for finitely many indices i , since $\ell_{\mathfrak{g}_A}(J^n I / J^{n+1} I) < \infty$), so $N^\bullet = \bigoplus_{n \geq 0} N^n$ is a $(S^\bullet(V), \mathfrak{g}_k)$ -module, which is finitely generated over $S^\bullet(V)$. We first have

$$\ell_{\mathfrak{g}_A}(J^n I / J^{n+1} I) = \ell_{\mathfrak{g}_R}(G_{J_m}^\bullet(J^n I / J^{n+1} I)),$$

hence the Hilbert series of the $(S^\bullet, \mathfrak{g}_A)$ -module $\bigoplus_{n \geq 0} J^n I / J^{n+1} I$ is the same as the Hilbert series of the $(S^\bullet, \mathfrak{g}_R)$ -module $\bigoplus_{n \geq 0} G_{J_m}^\bullet(J^n I / J^{n+1} I)$. Since $I \subset J_m$

and $J \subset J_m$ are inclusions of monomial ideals one can select a regular system of parameters $\{x_1, \dots, x_n\}$ of A as in Proposition 4.3, so that $J_m = (x_1, \dots, x_r)$, and I and J are generated by monomials in the parameters x_1, \dots, x_r (Prop. 4.2); note that the same regular system of parameters works for all monomial ideals simultaneously. Clearly, the R -module $G_{J_m}^i(J^n I / J^{n+1} I)$ is generated by elements $\{m_s\}_{s=1}^j$ where m_s is a product of monomial generators of I, J and J_m in the parameters x_1, \dots, x_r . Since $\alpha : \mathfrak{g}_R \rightarrow T_R$ is surjective (Prop. 4.3) there exist $\delta_1, \dots, \delta_{n-r} \in \mathfrak{g}_A \subset T_A(I)$ that map to elements \mathfrak{g}_R that lift the partial derivatives $\partial_{x_{r+1}}, \dots, \partial_{x_n} \in T_R$. Therefore $\delta_i = \partial_{x_{r+i}} + J_m \eta_i$, where $\eta_i \in T_A$. Now since $\mathbf{q}_A = \oplus A x_i \partial_{x_i}$ acts on \mathfrak{g}_A , a weight argument gives that $\partial_{x_{r+i}} \in \mathfrak{g}_A$. We can therefore assume that $\delta_i = \partial_{x_{r+i}}$, and thus $\delta_i \cdot m_s = 0$. Hence Proposition 2.15 implies that $G_{J_m}^i(J^n I / J^{n+1} I) \in \text{Loc}(\mathfrak{g}_R)$. Therefore the Hilbert series of the $(S^\bullet, \mathfrak{g}_R)$ -module $\oplus_{n \geq 0} G_{J_m}^\bullet(J^n I / J^{n+1} I)$ is the same as the Hilbert series of the $(S^\bullet(V), \mathfrak{g}_k)$ -module N^\bullet ,

$$H_{N^\bullet}(t) = \sum_{n \geq 0} \ell_{\mathfrak{g}_k}(N^n) t^n.$$

By Theorem 3.6 this is a rational function, and since the invariant ring $S^\bullet(V)^{n_k} = S^\bullet(V^{n_k})$ is generated in degree 1, so the exponents $n_i = 1$ in the rational expression for $H_{N^\bullet}(t)$, it follows in a standard way (see [20]) that the function $n \mapsto \chi_I^J(n)$ is given by a polynomial for high n . If $I = A$ and $J = J_m$, then $N^\bullet = S^\bullet(V)$, and $\ell_{\mathfrak{g}_k}(S^n(V)) = \dim S^n(V)^{n_k} = \dim S^n(V^{n_k})$, which implies that $d_{\mathfrak{g}_A}(A) = \ell_{\mathfrak{g}_k}(V)$ and $e_{\mathfrak{g}_A}(A, J_m) = 1$. \square

5. Complex analytic singularities

Let \mathcal{O}_n be the ring of complex convergent power series, and denote its maximal ideal \mathfrak{m} . Let $I \subset \mathfrak{m}$ be an ideal, $B = \mathcal{O}_n / I$, and $T_{\mathcal{O}_n}(I)$ the Lie algebroid of tangential derivations, so $T_B = T_{\mathcal{O}_n}(I) / IT_{\mathcal{O}_n}$. Let $J \subset \mathcal{O}_n$ be the contraction of the Jacobian ideal $\bar{J} \subset B$, as defined in Section 2.1, so for example when $I = (f)$, then $J = I + T_{\mathcal{O}_n} \cdot f$. Then $I \subset J$ is an inclusion of $T_{\mathcal{O}_n}(I)$ -preserved ideals (see [14, 26]). Put $A = \mathcal{O}_n / J$. There are surjective homomorphisms of \mathcal{O}_n -modules and Lie algebras $T_{\mathcal{O}_n}(I) \rightarrow T_B$ and $T_{\mathcal{O}_n}(J) \rightarrow T_A$. Note that a radical ideal I is an ideal of definition of the $T_{\mathcal{O}_n}(I)$ -module \mathcal{O}_n if and only if B is a regular local ring (Prop. 2.22).

It follows from Rossi's theorem [24] that we may assume (see e.g. a discussion in [7]), and we will do so for the remainder of this section, that

$$T_{\mathcal{O}_n}(I) \subset T_{\mathcal{O}_n}(\mathfrak{m}) = \mathfrak{m} T_{\mathcal{O}_n}.$$

This implies that \mathfrak{m} is the maximal defining ideal for any $T_{\mathcal{O}_n}(I)$ -module of finite type over \mathcal{O}_n , and the fibre $T_{\mathcal{O}_n}(I) / \mathfrak{m} T_{\mathcal{O}_n}(I)$ is a Lie algebra in a natural way.

Lemma 5.1. *Let $I \subset \mathfrak{m}^2$ be an ideal, and $J \subset \mathfrak{m}$ the Jacobian ideal of I . Consider \mathcal{O}_n as a $T_{\mathcal{O}_n}(I)$ -module. The following are equivalent:*

- (1) *J is an ideal of definition.*
- (2) $\dim_{\mathbb{C}} \mathcal{O}_n / J < \infty$.

(3) \mathcal{O}_n/I has an isolated singularity.

Proof. Put $\mathfrak{g} = T_{\mathcal{O}_n}(I)$. (1) \Rightarrow (2): By assumption, \mathfrak{m} is the maximal defining ideal of the \mathfrak{g} -module, hence $\sqrt{J} = \mathfrak{m}$ (Prop. 2.22), so there exists an integer r such that $\mathfrak{m}^r \subset J$. This implies (2). (2) \Rightarrow (1): This is obvious. (2) \Rightarrow (3): The ideal I defines a germ of an analytic variety $V \subset (\mathbb{C}^n, 0)$. (2) implies that $\sqrt{J} = \mathfrak{m}$, hence the (same noted) sheaf of ideal J on V has the property $J_{x'} = \mathcal{O}_{\mathbb{C}^n, x'}$ when $x' \neq 0$ is close to 0. By the Jacobian criterion of regularity, it follows that $\mathcal{O}_{\mathbb{C}^n, x'}/I_{x'}$ is a regular local ring.

(3) \Rightarrow (2): The locus $V(J)$ defines the singular locus of the variety germ $V(I) \subset (\mathbb{C}^n, 0)$. By assumption $V(J) = 0$, hence $\sqrt{J} = \mathfrak{m}$ by Hilbert's nullstellensatz for complex analytic spaces. This implies (2). \square

It is a fundamental problem to understand the structure of T_B -modules M such as B , $\bar{J} = BJ$ or $\bar{I}^c = BI^c$, where I^c is the integral closure of I , and for this purpose the Hilbert series $H_M(t)$ captures essential information. In the light of Proposition 3.15 it is important to know when the fibre Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of a Lie algebroid $\mathfrak{g} \subset T_{\mathcal{O}_n}$ is solvable.

Proposition 5.2. ([8]) Put $I = (f)$, $f \in \mathcal{O}_n$, and $\mathfrak{g} = T_{\mathcal{O}_n}(I)$. If \mathfrak{g} is free over \mathcal{O}_n and $n \leq 3$, then $\mathfrak{g}_{\mathbb{C}}$ is solvable.

Theorem 5.3. Let $I \subset \mathfrak{m}^2$ be an ideal generated by a regular sequence, and assume that \mathcal{O}_n/I has an isolated singularity. If I is a principal ideal, assume moreover that $I \subset \mathfrak{m}^3$. If $\mathfrak{g} = T_{\mathcal{O}_n}(I)$, then $\mathfrak{g}_{\mathbb{C}}$ is solvable.

Remark 5.4. Theorem 5.3 was proven in [7, Proposition 1.4] when I is a principal ideal, with a different method.

Proof. We have an injective homomorphism $T_{\mathcal{O}_n}(I)/IT_{\mathcal{O}_n} \rightarrow T_{\mathcal{O}_n}/IT_{\mathcal{O}_n}$, so we can identify $T_{\mathcal{O}_n}(I)/IT_{\mathcal{O}_n}$ with a submodule of $T_{\mathcal{O}_n}/IT_{\mathcal{O}_n}$. By the Artin-Rees lemma there exists a positive integer c such that for every integer $r > c + 1$, we have

$$(\mathfrak{m}^r \frac{T_{\mathcal{O}_n}}{IT_{\mathcal{O}_n}}) \cap \frac{T_{\mathcal{O}_n}(I)}{IT_{\mathcal{O}_n}} = \mathfrak{m}^{r-c}((\mathfrak{m}^c \frac{T_{\mathcal{O}_n}}{IT_{\mathcal{O}_n}}) \cap \frac{T_{\mathcal{O}_n}(I)}{IT_{\mathcal{O}_n}}) \subset \mathfrak{m}^2 \frac{T_{\mathcal{O}_n}(I)}{IT_{\mathcal{O}_n}},$$

and therefore $(I + \mathfrak{m}^r)T_{\mathcal{O}_n} \cap T_{\mathcal{O}_n}(I) \subset \mathfrak{m}^2 T_{\mathcal{O}_n}(I) + IT_{\mathcal{O}_n}$. Select r so that this inclusion holds. Since $T_{\mathcal{O}_n}(I) \subset T_{\mathcal{O}_n}(I + \mathfrak{m}^r)$ we have a canonical homomorphism of Lie algebroids

$$T_{\mathcal{O}_n}(I) \rightarrow T_{\mathcal{O}_n/(I + \mathfrak{m}^r)} = \frac{T_{\mathcal{O}_n}(I + \mathfrak{m}^r)}{(I + \mathfrak{m}^r)T_{\mathcal{O}_n}}.$$

The assumptions on I imply by [27, Korollar 3.2] that the finite-dimensional Lie algebra $T_{\mathcal{O}_n/(I + \mathfrak{m}^r)}$ is solvable, hence there exists a positive integer l such that

$$\mathcal{D}^l(T_{\mathcal{O}_n}(I)) \subset (I + \mathfrak{m}^r)T_{\mathcal{O}_n} \cap T_{\mathcal{O}_n}(I) \subset \mathfrak{m}^2 T_{\mathcal{O}_n}(I) + IT_{\mathcal{O}_n}.$$

Since $[\mathfrak{m}^2 T_{\mathcal{O}_n}(I) + IT_{\mathcal{O}_n}, \mathfrak{m}^2 T_{\mathcal{O}_n}(I) + IT_{\mathcal{O}_n}] \subset \mathfrak{m} T_{\mathcal{O}_n}(I)$ it follows that $\mathcal{D}^{l+1}(T_{\mathcal{O}_n}(I)) \subset \mathfrak{m} T_{\mathcal{O}_n}(I)$, and therefore $\mathcal{D}^{l+1}(\mathfrak{g}_{\mathbb{C}}) = 0$. \square

Theorem 5.5. Let $f \in \mathfrak{m}^3$, put $J = T_{\mathcal{O}_n} \cdot f + \mathcal{O}_n f$, and $\mathfrak{g} = T_{\mathcal{O}_n}(J)$. If $\mathcal{O}_n/(f)$ has an isolated singularity, then $\mathfrak{g}_{\mathbb{C}}$ is solvable.

Proof. Let N_s be the \mathcal{O}_n -submodule of $T_{\mathcal{O}_n}$ that $\mathcal{D}^s(T_{\mathcal{O}_n}(J))$ generates, $s = 0, 1, \dots$. By the Artin-Rees lemma there exists a positive integer $c = c(s)$ such that

$$N_s \cap \mathfrak{m}^l T_{\mathcal{O}_n} = \mathfrak{m}^{l-c}(\mathfrak{m}^c T_{\mathcal{O}_n} \cap N_s),$$

when $l > c$, implying $N_s \cap \mathfrak{m}^l T_{\mathcal{O}_n} \subset \mathfrak{m} T_{\mathcal{O}_n}(J)$ for such l . By [28], $T_{\mathcal{O}_n}(J)/JT_{\mathcal{O}_n}$ is solvable, hence $\mathcal{D}^s(T_{\mathcal{O}_n}(J)) \subset JT_{\mathcal{O}_n} \subset \mathfrak{m}^2 T_{\mathcal{O}_n}$, for sufficiently high s and since $f \in \mathfrak{m}^3$. Therefore there exists an integer $s' > s$ such that $\mathcal{D}^{s'}(T_{\mathcal{O}_n}(J)) \subset \mathfrak{m}^l T_{\mathcal{O}_n}$, implying

$$\mathcal{D}^{s'}(T_{\mathcal{O}_n}(J)) \subset \mathcal{D}^s(T_{\mathcal{O}_n}(J)) \cap \mathfrak{m}^l T_{\mathcal{O}_n} \subset N_s \cap \mathfrak{m}^l T_{\mathcal{O}_n} \subset \mathfrak{m} T_{\mathcal{O}_n}(J).$$

Hence $\mathcal{D}^{s'}(\mathfrak{g}_{\mathbf{C}}) = 0$. \square

Remark 5.6. The argument in [32] for the solvability of $T_{\mathcal{O}_n}(J)/JT_{\mathcal{O}_n}$ is incomplete, as noted in [28]. The proof in [28] on the other hand does depend on ideas from [32].

Theorem 5.7. For $f \in \mathfrak{m}$ we put $I_f = (f)$, $J_f = (f) + T_{\mathcal{O}_n} \cdot f$. Let $f, g \in \mathfrak{m}$ and assume that \mathcal{O}_n/I_f and \mathcal{O}_n/I_g both have an isolated singularity, and that their modular algebras are isomorphic, $\mathcal{O}_n/J_f \cong \mathcal{O}_n/J_g$. Then

$$H_{G_{J_f}(\mathcal{O}_n)}(t) = \sum_{i \geq 0} \dim_{\mathbf{C}} \frac{J_f^i}{J_f^{i+1}} t^i = \sum_{i \geq 0} \dim_{\mathbf{C}} \frac{J_g^i}{J_g^{i+1}} t^i.$$

Proof. According to Mather and Yau [18], if $\mathcal{O}_n/J_f \cong \mathcal{O}_n/J_g$, then there exists an isomorphism $\mathcal{O}_n \rightarrow \mathcal{O}_n$ mapping I_f onto I_g . Therefore J_f is mapped onto J_g , which implies the assertion. \square

Question 5.8. What is the dimension of the stratum of constant Hilbert series $H_{G_{J_f}(\mathcal{O}_n)}(t)$ in a semiversal deformation of a hypersurface with an isolated singularity?

Example 5.9. Assume that $I = (x_1, \dots, x_r) \subset \mathcal{O}_n$, where x_1, \dots, x_n is a regular system of parameters. Then $T_{\mathcal{O}_n}(I) = \sum_{i=r+1}^n \mathcal{O}_n \partial_{x_i} + \sum_{i,j=1}^r \mathcal{O}_n x_i \partial_{x_j}$ and the fibre Lie algebra

$$\mathfrak{g}_{\mathbf{C}} = \sum_{i=r+1}^n \mathbf{C} \partial_{x_i} + \sum_{i,j=1}^r \mathbf{C} x_i \partial_{x_j} = \sum_{i=r+1}^n \mathbf{C} \partial_{x_i} + \mathfrak{gl}_r$$

Example 5.10. Put $I = (\sum_{i=1}^n x_i^2) \subset \mathcal{O}_n$, $\partial_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i}$, and $E = \sum_{i=1}^n x_i \partial_{x_i}$. Then

$$T_{\mathcal{O}_n}(I) = \sum_{i < j} \mathcal{O}_n \partial_{ij} + \mathcal{O}_n E$$

and the fibre Lie algebra $\mathfrak{g}_{\mathbf{C}} = \mathbf{C} \oplus \mathfrak{o}_n$, where \mathfrak{o}_n is the orthogonal Lie algebra. This should not be a surprise!

Example 5.11. The Whitney umbrella is defined by the principal ideal $I = (z^2 - x^2 y) \subset \mathcal{O}_3$. We have $T_{\mathcal{O}_3}(I) = \sum_{i=1}^4 \mathcal{O}_3 \delta_i$, where

$$\begin{aligned} \delta_1 &= \nabla_x - 2\nabla_y \\ \delta_2 &= \nabla_x + \nabla_z \\ \delta_3 &= 2z\partial_y + x^2\partial_z \\ \delta_4 &= z\partial_x + xy\partial_z. \end{aligned}$$

The singularity is not isolated and $T_{\mathcal{O}_3}(I)$ is not free. The fibre Lie algebra $\mathfrak{g}_{\mathbf{C}} = \sum \mathbf{C}\bar{\delta}_i$, where $[\bar{\delta}_1, \bar{\delta}_2] = [\bar{\delta}_2, \bar{\delta}_4] = [\bar{\delta}_3, \bar{\delta}_4] = 0$ (note that $[\delta_3, \delta_4] = x\delta_1$), $[\bar{\delta}_1, \bar{\delta}_3] = 2[\bar{\delta}_2, \bar{\delta}_3] = 2\bar{\delta}_3$, and $[\bar{\delta}_1, \bar{\delta}_4] = -\bar{\delta}_4$. Therefore $\mathfrak{g}_{\mathbf{C}}$ is solvable.

Example 5.12. This example is borrowed from [8], where they start with a 4-dimensional representation of the Lie algebra $\mathfrak{gl}_2(\mathbf{C})$ and then apply Saito's criterion for free divisors to get the ideal $I = (y^2x^2 - 4xz^3 - 4y^3w + 18xyzw - 27w^2z^2) \subset \mathcal{O}_4$. Then $T_{\mathcal{O}_4}(I) = \oplus_{i=1}^4 \mathcal{O}_4\delta_i$, where

$$\begin{aligned}\delta_1 &= \nabla_x + \nabla_y + \nabla_z + \nabla_w \\ \delta_2 &= -3\nabla_x - \nabla_y + \nabla_z + 3\nabla_w \\ \delta_3 &= y\partial_x + 2z\partial_y + 3w\partial_z \\ \delta_4 &= 3x\partial_y + 2y\partial_z + z\partial_w.\end{aligned}$$

The ideal I is also seen to give the discriminant of a cubic polynomial. Here $n > 3$ and the singularity is not isolated, so we are not in the cases covered by Proposition 5.2 and Theorem 5.3. The fibre Lie algebra is $T_{\mathcal{O}_4}(I)/\mathfrak{m}T_{\mathcal{O}_4}(I) = \mathfrak{gl}_2(\mathbf{C})$, so a Levi factor is $\mathcal{L} = \mathfrak{sl}_2(\mathbf{C})$. The Hilbert series of the $T_{\mathcal{O}_4}(I)$ -module \mathcal{O}_4 is therefore the generating function for the numbers $\ell_{\mathfrak{sl}_2(\mathbf{C})}(S^n(\mathfrak{m}/\mathfrak{m}^2))$. Since $\mathfrak{m}/\mathfrak{m}^2$ is $\mathfrak{sl}_2(\mathbf{C})$ -simple of dimension 4, it can be identified with the symmetric product $V = S^3(\mathbf{C}^2)$. The Hilbert series $H_V(t)$ of the $(S^\bullet(V), \mathfrak{sl}_2(\mathbf{C}))$ -module $S^\bullet(V)$ is determined in Example 3.9. We get in particular the dimension $d_{T_{\mathcal{O}_4}(I)}(\mathcal{O}_4) = 2$ and multiplicity $e_{T_{\mathcal{O}_4}(I)}(\mathcal{O}_4, \mathfrak{m}) = 1/4$.

Question 5.13. Which finite-dimensional complex Lie algebras can be constructed as a fibre Lie algebra of a hypersurface singularity?

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